# Dominant strategies, ex post individual rationality, and optimal mechanisms*

Simon Loertscher† Junrok Park‡

November 9, 2026

**Abstract**

For a general mechanism design problem with independent private values, finitely many social alternatives and distributions satisfying regularity, we derive the optimal mechanisms subject to dominant strategy incentive compatibility and ex post individual rationality (IR). For any allocation rule implementable in dominant strategies, we characterize the additional expected rent required by ex post IR rather than interim IR and show that it vanishes iff a type is ex post worst off at almost every opponents' profile. For a two-agents partnership model, the optimal mechanisms are independent of the ownership structure, and commitment has no value for an agent whose payoff is being maximized.

**Keywords**: Mechanism design, contervailing incentives, non-equivalence

*We thank Leslie Marx and seminar participants at the National Taiwan University for valuable feedback. Financial support via a Visiting Research Scholar from the Faculty of Business and Economics at the University of Melbourne is also gratefully acknowledged.

†Department of Economics, Level 4, FBE Building, 111 Barry Street, University of Melbourne, Victoria 3010, Australia. Email: simonl@unimelb.edu.au.

‡National Taiwan University: Email: parkjunrok@ntu.edu.tw.

# 1 Introduction

Mechanism design theory is, arguably, one of the major achievements of economics over the past five to six decades. Of lonstanding interest within this theory is the question what a designer can obtain as a function of which notions of incentive compatibility and indivudal rationality are imposed. As a case in point, the motivation given by Myerson and Satterthwaite (1983) for their seminal paper was to prove that the impossibility result obtained by Vickrey (1961) for dominant strategies and ex post individual rationality carries over to Bayesian incentive compatibility and interim individual rationality. Remarkably, as subsequent work showed, for a large class of mechanism design setups with independent private values, the notions of incentive comatibility and individual rationality do *not* affect what a designer can achieve.[1]

This class of problems is such that incentive compatibility implies, for each agent, a *universal* worst-off type, that is, a type that is worst off for any allocation rule consistent with incentive compatibility. This means that incentive compatibility alone pins down where the individual rationality constraints will bind. Thereby, the class precludes problems with so-called *countervailing incentives*, which are such that the sets of worst-off types vary non-trivially with the allocation rule in the sense that there is no type that is worst off under any allocation rule consistent with incentive compatibility. Recent advances on multi-agent problems with countervailing incentives have brought to light a divergence of what can be achieved with dominant strategies and ex post individual rationality compared to any combination of other notions of incentive compatibility and indivudal rationality.[2] This raises the question of what are optimal mechanisms subject to dominant strategy incentive compatibility (DIC) and ex post individual rationaility (EIR) in problems with countervailing incentives.

This is the question we address in the present paper. We do so for a general class of mechanism design problems with finitely many social alternatives, lotteries over the alternatives, independent private values and type distributions satisfying regularity. The agents' payoffs are affine in their own type, with coefficients that can be positive or negative depending on the allocation. The marginal effect of type can therefore change sign across allocations, as in partnership dissolution, which serves as our running example. The designer maximizes a weighted sum of agents' expected utilities and expected profit. We analyze both the problem

[1] See, for example, Manelli and Vincent (2011) and the generalization by Gershkov et al. (2013).

[2] The term "countervailing incentives" was coined by Lewis and Sappington (1989). Early work on this family of problems, incldluing Maggi and Rodriguez-Clare (1995) and Jullien (2000), focused on single-agent problems as did Lewis and Sappington, in which case there is no difference in notions of incentive compatibility or individual rationality. Countervailing incentives problems with multiple agents are analyzed, for example, by Lu and Robert (2001), Loertscher and Wasser (2019) and Loertscher and Muir (2026).

with a fixed weight on profit, at least as large as each agent's welfare weight, and design subject to ex ante no deficit.

For a given allocation rule that is implementable in dominant strategies, we define an agent's *rent gap* as the difference between the minimum expected rents required by EIR and interim indivudal rationality. The gap is nonnegative and vanishes if and only if some type is ex post worst off at almost every profile of the other agents' types. Thus, the absence of such a *universal* worst-off type makes a positive rent gap possible; whether that gap arises depends on the allocation rule. We refer to problems without a type that is worst off under any allocation rule that is implementable in dominant strategies as problems with *countervailing incentives*.

We then derive the optimal mechanisms subject to DIC and EIR using the sign-dependent virtual-utility approach of Andreyanov et al. (2023). An allocation with a positive own-type coefficient is evaluated at a virtual value, whereas one with a negative coefficient is evaluated at a virtual cost. Under regularity, both virtual values and virtual costs are strictly increasing. Consequently, a higher type makes allocations with larger own-type coefficients relatively more attractive, so pointwise virtual-surplus maximization produces a monotone allocation rule. Setting each agent's payoff to zero at an ex post worst-off type then yields an optimal DIC–EIR mechanism. The same construction accommodates agents who sometimes buy and sometimes sell.

Partnership models are prominent models exhibiting coutnervailing incentives and are nested in our general framework. Under interim individual rationality, the optimal mechanisms vary with the ownship structure. In contrast, under EIR the optimal mechanisms and the virtual type comparisons are independent of the ownership structure, which only affects the quantities traded. The model thus recovers the bilateral trade problems of Myerson and Satterthwaite (1983) and Williams (1987) as special cases with extreme ownership where in one instance the quantity traded is simply zero because the agent who is supposed to sell has nothing to sell.

The ex post constraints of DIC and EIR also have implications for the timing of contracting. Suppose one partner chooses the mechanism to maximize his own expected utility subject to DIC, EIR, and ex ante no deficit. We show that he obtains the same ex ante value whether he contracts before learning his type or after his type becomes publicly known. An optimal ex ante mechanism can be chosen so that its expected budget balances conditional on every realization of his type and its conditional contracts remain optimal after disclosure. Thus, a single ex ante contract attains the optimum in every public-disclosure state. In contrast, the two-partner uniform example has a positive ex ante commitment premium under interior ownership with interim individual rationality (Mylovanov and Tröger, 2014).

Our analysis builds on the participation comparisons of Makowski and Mezzetti (1994) and Loertscher and Muir (2026). The former study first-best implementation under interim and ex post participation, while the latter identify the revenue cost of imposing EIR on an interim-optimal allocation in Hotelling auctions. We characterize this cost for arbitrary DIC allocation rules in our environment, incorporate allocation reoptimization, and use the comparison to establish strict Pareto improvements from inefficient benchmarks. Kosmopoulou (1999) provides a sufficient condition for the first-best participation gap to vanish. The separate incentive and participation equivalences of Gershkov et al. (2013) and Chen et al. (2019) provide the benchmarks against which the joint restriction is assessed.

The dependence of participation rents on worst-off types connects our analysis to the literature on countervailing incentives, including Lewis and Sappington (1989), Maggi and Rodriguez-Clare (1995), Jullien (2000), Lu and Robert (2001), Brocas (2013), and Loertscher and Wasser (2019). Building on the IR critical-type saddle-point results of Loertscher and Wasser (2019) and Loertscher and Muir (2026), we extend the approach to DIC–EIR, where critical types may depend on opponents' reports. The appendix establishes attainment and the saddle-point characterization for this extension, complementing the ex post virtual-utility approach of Andreyanov et al. (2023). Our ownership comparison contributes to the partnership-dissolution literature initiated by Cramton et al. (1987), with subsequent contributions including Fieseler et al. (2003), Galavotti et al. (2011), and Loertscher and Wasser (2019). Our contracting comparison concerns the ex ante and public-disclosure regimes studied by Mylovanov and Tröger (2014).

The remainder of the paper is organized as follows. Section 2 introduces the environment and the separate equivalences. Section 3 characterizes rent gaps and optimal DIC–EIR mechanisms. Section 4 establishes the strict Pareto comparison, and Section 5 studies public disclosure of a participant's type. Section 6 specializes the general results to partnership dissolution.

# 2 The Model

We consider a finite social-choice environment that allows countervailing incentives.

## 2.1 Primitives and Mechanisms

There is a finite set of agents $\mathcal{N} = \{1, \dots, n\}$. Each agent $i$ privately observes his type $\theta_i \in [0, 1]$. Types are independently drawn from distributions $F_i$ with densities $f_i$ bounded away from zero on $[0, 1]$. Write $F := \prod_i F_i$ for the joint distribution, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ for the

type profile, and $\boldsymbol{\theta}_{-i}$ for the profile of types other than $i$'s.

Let $\mathcal{K} := \{1, \ldots, K\}$ be a finite set of deterministic assignments, or *social alternatives*. The designer can commit to lotteries over $\mathcal{K}$. An allocation $\boldsymbol{q} := (q_k)_{k \in \mathcal{K}} \in \Delta(\mathcal{K})$ specifies the probability $q_k$ of each assignment $k$.

For each assignment $k$, agent $i$'s gross payoff is $a_i^k + \theta_i m_i^k$, where $a_i^k$ is his *type-independent payoff component* and $m_i^k$ his *own-type coefficient*. Agents are risk neutral, so gross payoff under an allocation $\boldsymbol{q}$ is

$$a_i(\boldsymbol{q}) + \theta_i m_i(\boldsymbol{q}), \qquad a_i(\boldsymbol{q}) := \sum_{k \in \mathcal{K}} q_k a_i^k, \quad m_i(\boldsymbol{q}) := \sum_{k \in \mathcal{K}} q_k m_i^k.$$

For a fixed allocation $\boldsymbol{q}$, gross payoff increases with type when $m_i(\boldsymbol{q}) > 0$, decreases when $m_i(\boldsymbol{q}) < 0$, and is independent of type otherwise. Values are private: other agents' types affect this payoff only through the selected allocation.

The designer incurs a cost $C^k \geq 0$ under assignment $k$. Under allocation $\boldsymbol{q}$, the expected cost is $C(\boldsymbol{q}) := \sum_{k \in \mathcal{K}} q_k C^k$.

Each agent may refuse the mechanism and receive a payoff normalized to zero. We assume that a feasible *status-quo allocation* $\boldsymbol{q}^0 \in \Delta(\mathcal{K})$, together with zero transfers, delivers this payoff to every type and entails no cost: $a_i(\boldsymbol{q}^0) = m_i(\boldsymbol{q}^0) = 0$ for every $i$, and $C(\boldsymbol{q}^0) = 0$.

By the revelation principle, it is without loss to restrict attention to direct mechanisms. A direct mechanism $\langle \boldsymbol{q}, \boldsymbol{t} \rangle$ consists of a measurable allocation rule $\boldsymbol{q} : [0,1]^n \to \Delta(\mathcal{K})$ and measurable transfer rules $t_i : [0,1]^n \to \mathbb{R}$, where $t_i$ is the payment made by agent $i$ to the designer.[3] If agent $i$ has type $\theta_i$, reports $\hat{\theta}_i$, and the other agents report $\boldsymbol{\theta}_{-i}$, his payoff is

$$U_i(\theta_i, \hat{\theta}_i, \boldsymbol{\theta}_{-i}) := a_i(\boldsymbol{q}(\hat{\theta}_i, \boldsymbol{\theta}_{-i})) + \theta_i m_i(\boldsymbol{q}(\hat{\theta}_i, \boldsymbol{\theta}_{-i})) - t_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i}).$$

His truthful payoff is $U_i(\boldsymbol{\theta}) := U_i(\theta_i, \theta_i, \boldsymbol{\theta}_{-i})$.

In partnership dissolution, $n \geq 2$ partners initially own shares $r_i \in (0,1)$, with $\sum_i r_i = 1$, of a single asset. Partner $i$ values the entire asset at $\theta_i$. The alternatives are $\mathcal{K} = \mathcal{N}$, with alternative $k$ assigning the entire asset to partner $k$, and allocation costs are zero. Thus $q_i$ is the probability that partner $i$ receives the asset, and his expected payoff relative to initial ownership is $(q_i - r_i)\theta_i - t_i$. In the general notation, $a_i^k = 0$, $m_i^k = \mathbf{1}\{i = k\} - r_i$, and $C^k = 0$; the status quo is $\boldsymbol{q}^0 = (r_1, \ldots, r_n)$. Each partner has a positive own-type coefficient when he acquires the asset and a negative coefficient when another partner acquires it. Section 6 specializes this setup to two partners with uniform types.

[3] Throughout, mechanisms and conditional contracts are measurable, and transfers and induced utilities have finite interim, ex ante, and conditional expectations wherever these are used below.

## 2.2 Incentives and Participation

We distinguish two incentive requirements and two participation requirements. A mechanism is Bayesian incentive compatible (BIC) if truthful reporting maximizes each agent's interim expected payoff, and dominant-strategy incentive compatible (DIC) if truthful reporting maximizes his payoff for every profile of the other agents' reports:

$$\mathbb{E}_{-i}[U_i(\boldsymbol{\theta})] \geq \mathbb{E}_{-i}[U_i(\theta_i, \hat{\theta}_i, \boldsymbol{\theta}_{-i})], \qquad \forall i,\ \theta_i, \hat{\theta}_i \in [0,1], \tag{BIC}$$

$$U_i(\boldsymbol{\theta}) \geq U_i(\theta_i, \hat{\theta}_i, \boldsymbol{\theta}_{-i}), \qquad \forall i,\ \theta_i, \hat{\theta}_i \in [0,1],\ \boldsymbol{\theta}_{-i} \in [0,1]^{n-1}, \tag{DIC}$$

A mechanism is interim individually rational (IR) if every type obtains a nonnegative interim expected payoff, and ex post individually rational (EIR) if every agent obtains a nonnegative payoff at every type profile:

$$\mathbb{E}_{-i}[U_i(\boldsymbol{\theta})] \geq 0, \qquad \forall i,\ \theta_i \in [0,1], \tag{IR}$$

$$U_i(\boldsymbol{\theta}) \geq 0, \qquad \forall i,\ \boldsymbol{\theta} \in [0,1]^n. \tag{EIR}$$

Here, $\mathbb{E}_{-i}$ denotes expectations over the other agents' types under $F_{-i} := \prod_{j \neq i} F_j$.

An allocation rule is BIC-implementable or DIC-implementable if some transfer rules make the resulting mechanism satisfy the corresponding incentive requirement. For a feasible allocation rule $\boldsymbol{q}$, define agent $i$'s *interim own-type coefficient* by

$$M_i^{\boldsymbol{q}}(\theta_i) := \mathbb{E}_{-i}\left[m_i(\boldsymbol{q}(\theta_i, \boldsymbol{\theta}_{-i}))\right].$$

The rule is BIC-implementable if and only if $M_i^{\boldsymbol{q}}$ is weakly increasing for every agent. It is DIC-implementable if and only if

$$\theta_i \longmapsto m_i(\boldsymbol{q}(\theta_i, \boldsymbol{\theta}_{-i}))$$

is weakly increasing for every agent $i$ and every opponents' report profile $\boldsymbol{\theta}_{-i}$ (Börgers, 2015, Sections 5.6, 6.3, and 7.2). Absorbing $a_i(\boldsymbol{q}(\boldsymbol{\theta}))$ into transfers yields the standard single-parameter problem, evaluated in expectation under BIC and at each opponents' report profile under DIC. Let $\mathcal{Q}^{\text{BIC}}$ and $\mathcal{Q}^{\text{DIC}}$ denote the respective classes of feasible allocation rules. Participation and budget requirements are imposed separately.

## 2.3 The Design Problem

The designer's expected profit is

$$\Pi(\boldsymbol{q}, \boldsymbol{t}) := \sum_{i=1}^{n} \mathbb{E}[t_i(\boldsymbol{\theta})] - \mathbb{E}[C(\boldsymbol{q}(\boldsymbol{\theta}))].$$

The designer puts welfare weight $w_i \geq 0$ on agent $i$'s expected payoff and a fixed weight $\kappa > 0$ on expected profit. The objective is

$$W_\kappa(\boldsymbol{q}, \boldsymbol{t}) := \sum_{i=1}^{n} w_i \mathbb{E}[U_i(\boldsymbol{\theta})] + \kappa \Pi(\boldsymbol{q}, \boldsymbol{t}).$$

For $X \in \{\text{BIC}, \text{DIC}\}$ and $Y \in \{\text{IR}, \text{EIR}\}$, let $V_{X-Y}(\kappa)$ be the supremum of $W_\kappa$ over mechanisms satisfying $X$ and $Y$. We call this the *fixed-$\kappa$ problem*. This problem imposes no budget constraint.

We rewrite the designer's objective in terms of expected total surplus and agents' expected payoffs. Fix an allocation $\boldsymbol{q} \in \Delta(\mathcal{K})$ and a type profile $\boldsymbol{\theta}$, define total surplus by

$$s(\boldsymbol{q}, \boldsymbol{\theta}) := \sum_{i=1}^{n} [a_i(\boldsymbol{q}) + \theta_i m_i(\boldsymbol{q})] - C(\boldsymbol{q}).$$

Let $S(\boldsymbol{q}) := \mathbb{E}[s(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{\theta})]$ denote expected total surplus. Transfers redistribute surplus between the agents and the designer, so

$$S(\boldsymbol{q}) = \sum_{i=1}^{n} \mathbb{E}[U_i(\boldsymbol{\theta})] + \Pi(\boldsymbol{q}, \boldsymbol{t}).$$

With $\alpha_i := (\kappa - w_i)/\kappa$, the objective is therefore

$$W_\kappa(\boldsymbol{q}, \boldsymbol{t}) = \kappa \left[ S(\boldsymbol{q}) - \sum_{i=1}^{n} \alpha_i \mathbb{E}[U_i(\boldsymbol{\theta})] \right].$$

Holding the allocation fixed, an additional dollar of expected payoff for agent $i$ reduces expected profit by one dollar. Its net cost to the objective is $\kappa - w_i$, so $\alpha_i$ measures this cost relative to the weight on profit. We restrict the fixed-$\kappa$ problem to $\kappa \geq \max_{i \in \mathcal{N}} w_i$, so $\alpha_i \geq 0$.[4]

We also study design problems that require the designer to avoid an expected deficit. A

[4]Without a budget constraint, if $w_i > \kappa$, a report-independent subsidy to agent $i$ preserves incentive compatibility and participation while raising $W_\kappa$ without bound.

mechanism satisfies *ex ante no deficit* (EAND) if

$$\Pi(\boldsymbol{q}, \boldsymbol{t}) \geq 0. \tag{EAND}$$

Let $w_0 \geq 0$ be the baseline weight on expected profit, with $\max\{w_0, w_1, \ldots, w_n\} > 0$. The objective in the EAND-constrained problem is

$$J(\boldsymbol{q}, \boldsymbol{t}) := \sum_{i=1}^{n} w_i \mathbb{E}[U_i(\boldsymbol{\theta})] + w_0 \Pi(\boldsymbol{q}, \boldsymbol{t}).$$

For a multiplier $\lambda \geq 0$ on the EAND constraint, $J(\boldsymbol{q}, \boldsymbol{t}) + \lambda\Pi(\boldsymbol{q}, \boldsymbol{t}) = W_{w_0+\lambda}(\boldsymbol{q}, \boldsymbol{t})$. Thus, the fixed-$\kappa$ objective is the Lagrangian objective for the EAND-constrained problem when $\kappa = w_0 + \lambda$. The effective weight on profit includes both its baseline weight and the multiplier on the budget constraint.

Strengthening incentive compatibility or participation separately preserves the designer's value in this environment.

**Lemma 1** (Separate robustness equivalences)**.** *Fix $\kappa > 0$ with $\alpha_i \geq 0$ for every $i$. Then*

$$V_{\mathrm{BIC-IR}}(\kappa) = V_{\mathrm{BIC-EIR}}(\kappa) = V_{\mathrm{DIC-IR}}(\kappa).$$

The BIC–IR to DIC–IR comparison follows from the BIC–DIC equivalence of Gershkov et al. (2013), adapted to allow own-type coefficients of either sign. The BIC–IR to BIC–EIR comparison uses the transfer adjustment discussed by Chen et al. (2019, Remark 6). Both constructions preserve interim utilities and expected profit, and hence $W_\kappa$. The appendix provides the details. We ask whether imposing DIC and EIR jointly also preserves the designer's value.

# 3 Rent Gaps and Optimal Design

The separate equivalences leave open whether imposing DIC and EIR jointly preserves the designer's value. We first compare the minimum participation rents at a fixed DIC allocation rule and characterize their difference through worst-off types. We then relate this rent gap to countervailing incentives in the environment and use virtual surplus to characterize optimal DIC–EIR mechanisms.

## 3.1 Worst-Off Types and Rent Gaps

Worst-off types determine the payoff levels required for participation. For a fixed allocation rule, the envelope formulas determine payoff differences across an agent's types, while participation requires the lowest payoff to be nonnegative.

For a feasible allocation rule, define agent $i$'s *accumulated information rent* by

$$R_i^{\boldsymbol{q}}(x, \boldsymbol{\theta}_{-i}) := \int_0^x m_i(\boldsymbol{q}(s, \boldsymbol{\theta}_{-i}))\, ds.$$

This function records the accumulated effect of types on payoff; it does not fix the agent's payoff level. Under DIC, the envelope formula gives

$$U_i(\theta_i, \boldsymbol{\theta}_{-i}) - U_i(x, \boldsymbol{\theta}_{-i}) = R_i^{\boldsymbol{q}}(\theta_i, \boldsymbol{\theta}_{-i}) - R_i^{\boldsymbol{q}}(x, \boldsymbol{\theta}_{-i})$$

for every agent $i$, all $\theta_i, x \in [0,1]$, and every opponents' report profile. Under BIC, the corresponding identity holds in expectation:

$$\mathbb{E}_{-i}[U_i(\theta_i, \boldsymbol{\theta}_{-i})] - \mathbb{E}_{-i}[U_i(x, \boldsymbol{\theta}_{-i})] = \mathbb{E}_{-i}[R_i^{\boldsymbol{q}}(\theta_i, \boldsymbol{\theta}_{-i}) - R_i^{\boldsymbol{q}}(x, \boldsymbol{\theta}_{-i})].$$

Thus, incentive compatibility determines interim payoffs up to one constant per agent under BIC, and pointwise payoffs up to one constant per opponents' report profile under DIC.

For $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{BIC}}$, define agent $i$'s set of *interim worst-off types* by

$$\Omega_i(\boldsymbol{q}) := \arg\min_{x \in [0,1]} \mathbb{E}_{-i}[R_i^{\boldsymbol{q}}(x, \boldsymbol{\theta}_{-i})].$$

IR requires nonnegative interim payoff at these types. For $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}$, ex post worst-off types minimize $R_i^{\boldsymbol{q}}(\cdot, \boldsymbol{\theta}_{-i})$ at a given profile $\boldsymbol{\theta}_{-i}$. EIR requires nonnegative payoff at these minimizing types for every profile of the other agents' reports. Because the finite collection of own-type coefficients is bounded, $R_i^{\boldsymbol{q}}(\cdot, \boldsymbol{\theta}_{-i})$ is Lipschitz continuous with a bound uniform across opponents' profiles. Its expectation is therefore also Lipschitz continuous. Both attain their minima on the compact interval $[0,1]$, so the interim and each profilewise ex post minimizing set are nonempty.

To compare participation costs for a fixed DIC allocation rule $\boldsymbol{q}$, write

$$U_i(\theta_i, \boldsymbol{\theta}_{-i}) = h_i(\boldsymbol{\theta}_{-i}) + R_i^{\boldsymbol{q}}(\theta_i, \boldsymbol{\theta}_{-i}).$$

Under IR, setting the minimum interim utility to zero determines $\mathbb{E}_{-i}[h_i]$; under EIR, setting

the minimum utility to zero at each opponents' report profile determines $h_i$ itself:

$$\begin{aligned} \text{IR:} \quad & \mathbb{E}_{-i}[h_i(\boldsymbol{\theta}_{-i})] = -\min_{x\in[0,1]} \mathbb{E}_{-i}[R_i^{\boldsymbol{q}}(x, \boldsymbol{\theta}_{-i})], \\ \text{EIR:} \quad & h_i(\boldsymbol{\theta}_{-i}) = -\min_{x\in[0,1]} R_i^{\boldsymbol{q}}(x, \boldsymbol{\theta}_{-i}). \end{aligned} \tag{1}$$

We call these choices *worst-off normalizations*: they minimize expected utility subject to IR or EIR while preserving DIC.

Under either normalization, expected payoff equals $\mathbb{E}[R_i^{\boldsymbol{q}}(\theta_i, \boldsymbol{\theta}_{-i})] + \mathbb{E}_{-i}[h_i(\boldsymbol{\theta}_{-i})]$. Subtracting the IR minimum from the EIR minimum cancels the common accumulated-rent term, yielding the rent gap.

**Definition 1** (Rent gap)**.** *For $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}$, define agent $i$'s* rent gap *by*

$$\Gamma_i(\boldsymbol{q}) := \min_{x\in[0,1]} \mathbb{E}_{-i}[R_i^{\boldsymbol{q}}(x, \boldsymbol{\theta}_{-i})] - \mathbb{E}_{-i}\left[\min_{x\in[0,1]} R_i^{\boldsymbol{q}}(x, \boldsymbol{\theta}_{-i})\right].$$

To characterize when the two minimum expected rents coincide, we ask whether a single type minimizes accumulated rent at almost every opponents' profile. For $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}$, define the set of *common ex post worst-off types* by

$$\Omega_i^{EP}(\boldsymbol{q}) := \left\{ x \in [0,1] : x \in \arg\min_{z\in[0,1]} R_i^{\boldsymbol{q}}(z, \boldsymbol{\theta}_{-i}) \text{ for } F_{-i}\text{-almost every } \boldsymbol{\theta}_{-i} \right\}.$$

The common set can be empty even though each profilewise minimizing set is nonempty.[5]

The characterization is as follows.

**Theorem 1** (Rent-gap characterization)**.** *Fix $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}$. For every agent $i$, $\Gamma_i(\boldsymbol{q}) \geq 0$, and the minimum expected rent among DIC–EIR implementations of $\boldsymbol{q}$ equals the minimum among DIC–IR implementations plus $\Gamma_i(\boldsymbol{q})$. Moreover,*

$$\Gamma_i(\boldsymbol{q}) = 0 \iff \Omega_i^{EP}(\boldsymbol{q}) \neq \varnothing \iff \Omega_i^{EP}(\boldsymbol{q}) = \Omega_i(\boldsymbol{q}).$$

*Proof.* Fix $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}$ and an agent $i$. The worst-off normalizations in (1) attain the minimum expected rents under IR and EIR; subtracting these rents gives $\Gamma_i(\boldsymbol{q})$. Appendix A.2.2 verifies that these normalizations yield measurable, integrable payoffs and transfers. For any $x_i^* \in \Omega_i(\boldsymbol{q})$,

$$\Gamma_i(\boldsymbol{q}) = \mathbb{E}_{-i}\left[R_i^{\boldsymbol{q}}(x_i^*, \boldsymbol{\theta}_{-i}) - \min_{x\in[0,1]} R_i^{\boldsymbol{q}}(x, \boldsymbol{\theta}_{-i})\right].$$

[5] The almost-everywhere qualifier concerns commonality across reports; DIC and EIR remain requirements at every report profile.

The integrand is nonnegative, so $\Gamma_i(\boldsymbol{q}) \geq 0$, with equality if and only if $x_i^*$ minimizes accumulated rent at almost every opponents' profile. Thus a zero rent gap makes every interim worst-off type a common ex post worst-off type. Conversely, a common ex post worst-off type also minimizes expected accumulated rent and makes the integrand zero almost surely. Since $\Omega_i(\boldsymbol{q})$ is nonempty and every common ex post worst-off type is interim worst off, the three conditions are equivalent. EIR therefore requires strictly more expected rent precisely when no type is worst off at almost every opponents' profile. □

## 3.2 Countervailing Incentives

Section 3.1 asks whether a fixed allocation rule has a worst-off type common across opponents' reports. We now ask whether one type remains worst off across all DIC allocation rules. Define the set of *universal ex post worst-off types* by

$$\Omega_i^{EP} := \bigcap_{\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}} \Omega_i^{EP}(\boldsymbol{q})$$

**Definition 2** (Standard problems and countervailing incentives)**.** *Agent $i$* faces countervailing incentives *if $\Omega_i^{EP} = \varnothing$. A mechanism-design problem is* standard *if $\Omega_i^{EP} \neq \varnothing$ for every agent $i$. Otherwise, the problem exhibits* countervailing incentives.

The following lemma characterizes countervailing incentives in terms of own-type coefficients and shows that intersecting interim worst-off sets over all BIC allocation rules yields the same universal set.

**Lemma 2** (Primitive characterization of countervailing incentives)**.** *For every agent $i$, $\Omega_i^{EP} = \bigcap_{\boldsymbol{q} \in \mathcal{Q}^{\mathrm{BIC}}} \Omega_i(\boldsymbol{q})$. A mechanism-design problem exhibits countervailing incentives if and only if some agent $i$ satisfies*

$$\min_{k \in \mathcal{K}} m_i^k < 0 < \max_{k \in \mathcal{K}} m_i^k.$$

If an agent's own-type coefficients are all nonnegative, type 0 is universally worst off, as in a sale problem. If they are all nonpositive, type 1 is universally worst off, as in a procurement problem. When the coefficients take both signs, constant allocation rules selecting a negative or a positive coefficient make types 1 and 0, respectively, uniquely worst off, leaving no universal worst-off type.[6]

[6]The sign characterization connects the definition to the countervailing-incentive phenomenon studied by Lewis and Sappington (1989). They describe countervailing incentives as arising when an agent prefers to understate his private information for some realizations and to overstate it for others. Here the primitive source is that an agent's marginal payoff from type can have either sign across assignments.

This classification has a direct implication for participation costs. An agent with a universal ex post worst-off type has $\Gamma_i(\boldsymbol{q}) = 0$ under every DIC allocation rule. Hence all rent gaps vanish in a standard problem.

Countervailing incentives in the environment do not imply a positive rent gap for every DIC allocation rule. The classification concerns the absence of a universal worst-off type across all such rules, whereas the rent gap concerns one fixed rule. Under the constant status-quo rule, $m_i(\boldsymbol{q}^0) = 0$ for every agent, so accumulated rent is identically zero, $\Omega_i^{EP}(\boldsymbol{q}^0) = [0, 1]$, and $\Gamma_i(\boldsymbol{q}^0) = 0$.

## 3.3 Virtual Surplus

The preceding analysis identifies minimum participation rents. We now use the EIR normalization to express the designer's objective directly in terms of allocations. Virtual values and costs incorporate these rents into the evaluation of own-type coefficients. For each agent $i$, define the *virtual value* and *virtual cost*, respectively, by

$$\psi_i^B(\theta_i) := \theta_i - \alpha_i \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}, \qquad \psi_i^S(\theta_i) := \theta_i + \alpha_i \frac{F_i(\theta_i)}{f_i(\theta_i)}.$$

Because $\alpha_i \geq 0$, virtual value lies weakly below the agent's type, while virtual cost lies weakly above it. Both coincide with the type when $\alpha_i = 0$.

For an allocation $\boldsymbol{x} \in \Delta(\mathcal{K})$ and a type profile $\boldsymbol{\theta}$, define the *pointwise virtual surplus* by

$$\Phi_\kappa(\boldsymbol{x}, \boldsymbol{\theta}) := \sum_{i=1}^{n} \left[ a_i(\boldsymbol{x}) + \psi_i^B(\theta_i) m_i^+(\boldsymbol{x}) - \psi_i^S(\theta_i) m_i^-(\boldsymbol{x}) \right] - C(\boldsymbol{x}),$$

where, for any $z \in \mathbb{R}$, $z^+ := \max\{z, 0\}$ and $z^- := \max\{-z, 0\}$. The subscript records the dependence on $\kappa$ through $\alpha_i$.

Now fix an allocation rule $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}$ and implement it with the EIR worst-off normalization in (1). By DIC monotonicity, a profilewise worst-off type separates nonpositive from nonnegative own-type coefficients. Integrating the normalized payoff over own type therefore evaluates a positive own-type coefficient at virtual value and a negative own-type coefficient at virtual cost, almost surely. With EIR-normalized transfers $\boldsymbol{t}$,

$$W_\kappa(\boldsymbol{q}, \boldsymbol{t}) = \kappa \, \mathbb{E}\left[ \Phi_\kappa(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{\theta}) \right]. \tag{2}$$

For DIC–EIR, the pointwise representation in (2) provides a direct route to optimization. Maximizing $\Phi_\kappa$ separately at each type profile gives an upper bound on the objective; at-

taining this bound requires the resulting allocation rule to be DIC-implementable.

For comparison, the BIC–IR objective admits a virtual-surplus representation that depends on the interim allocation rule. Whereas $\Phi_\kappa(\boldsymbol{x}, \boldsymbol{\theta})$ evaluates a single allocation $\boldsymbol{x}$, $\Phi_\kappa^{IR}(\boldsymbol{q}, \boldsymbol{\theta})$ depends on the rule $\boldsymbol{q}$ through its interim own-type coefficients $M_i^{\boldsymbol{q}}(\theta_i)$. Under EIR, virtual values and costs apply to the positive and negative parts of $m_i(\boldsymbol{q}(\boldsymbol{\theta}))$ at each profile. Under IR, they apply to the positive and negative parts of its interim average, $M_i^{\boldsymbol{q}}(\theta_i)$. Specifically, for any BIC allocation rule $\boldsymbol{q}$, the objective under minimum-rent IR transfers equals $\kappa \mathbb{E}[\Phi_\kappa^{IR}(\boldsymbol{q}, \boldsymbol{\theta})]$, where

$$\Phi_\kappa^{IR}(\boldsymbol{q}, \boldsymbol{\theta}) := \sum_{i=1}^{n} \Big[ a_i(\boldsymbol{q}(\boldsymbol{\theta})) + \psi_i^B(\theta_i)\big(M_i^{\boldsymbol{q}}(\theta_i)\big)^+ - \psi_i^S(\theta_i)\big(M_i^{\boldsymbol{q}}(\theta_i)\big)^- \Big] - C(\boldsymbol{q}(\boldsymbol{\theta})). \tag{3}$$

Averaging before taking these parts allows positive and negative coefficients across opponents' profiles to offset one another. Although restricting the search to DIC allocation rules preserves the BIC–IR value, these interim averages still link allocations across profiles. The IR representation therefore does not directly reduce the allocation problem to pointwise maximization.[7] Proposition 1 in Appendix A.2.2 establishes the fixed-rule IR representation. Together with Lemma 1, it permits restriction to DIC rules without changing the IR value.

Returning to DIC–EIR, we impose two-sided regularity to ensure that pointwise virtual-surplus maximization yields a DIC-implementable allocation rule.

**Assumption 1** (Two-sided regularity)**.** *For every agent $i$, the density $f_i$ is continuous and strictly positive on $[0, 1]$, $(1 - F_i)/f_i$ is weakly decreasing, and $F_i/f_i$ is weakly increasing.*

This assumption makes virtual values and costs continuous and strictly increasing. These properties ensure that every measurable virtual-surplus-maximizing selection has weakly increasing own-type coefficients, as the next subsection establishes.

## 3.4 Optimality and Value

We now show that a DIC–EIR mechanism attains the pointwise virtual-surplus bound derived above. We then compare the optimized DIC–EIR and BIC–IR values, using the rent gap to identify the gains from relaxing the constraints.

To construct the candidate mechanism, choose a measurable pointwise virtual-surplus

[7] To solve the IR virtual surplus maximization problem $\max_{\boldsymbol{q} \in \mathcal{Q}^{\text{BIC}}} \kappa \mathbb{E}\left[\Phi_\kappa^{IR}(\boldsymbol{q}, \boldsymbol{\theta})\right]$, one may select a profile of interim allocation rules subject to an extension of the interim feasibility condition due to Border (2007).

maximizer at every type profile:

$$\boldsymbol{q}^{VSM}(\boldsymbol{\theta}) \in \arg\max_{\boldsymbol{x}\in\Delta(\mathcal{K})} \Phi_\kappa(\boldsymbol{x},\boldsymbol{\theta}). \tag{4}$$

We call $\boldsymbol{q}^{VSM}$ the *VSM allocation rule*. Apply the EIR worst-off normalization and set transfers by

$$\begin{aligned} U_i^{VSM}(\boldsymbol{\theta}) &:= R_i^{\boldsymbol{q}^{VSM}}(\theta_i,\boldsymbol{\theta}_{-i}) - \min_{x\in[0,1]} R_i^{\boldsymbol{q}^{VSM}}(x,\boldsymbol{\theta}_{-i}),\\ t_i^{VSM}(\boldsymbol{\theta}) &:= a_i(\boldsymbol{q}^{VSM}(\boldsymbol{\theta})) + \theta_i m_i(\boldsymbol{q}^{VSM}(\boldsymbol{\theta})) - U_i^{VSM}(\boldsymbol{\theta}). \end{aligned}$$

The pair $(\boldsymbol{q}^{VSM},\boldsymbol{t}^{VSM})$ is the *VSM mechanism*.

**Theorem 2** (VSM optimality and value comparison)**.** *Maintain Assumption 1 and fix $\kappa>0$ with $\alpha_i\geq 0$ for every $i$.*

(i) *The VSM mechanism $(\boldsymbol{q}^{VSM},\boldsymbol{t}^{VSM})$ constructed above satisfies DIC and EIR and solves the fixed-$\kappa$ DIC–EIR problem. Conversely, every optimal mechanism has an allocation rule $\boldsymbol{q}$ such that $\boldsymbol{q}(\boldsymbol{\theta})$ maximizes $\Phi_\kappa(\cdot,\boldsymbol{\theta})$ for $F$-almost every $\boldsymbol{\theta}$.*

(ii) *The optimal values satisfy*

$$\begin{aligned} V_{\mathrm{DIC-EIR}}(\kappa) &= \max_{\boldsymbol{q}\in\mathcal{Q}^{\mathrm{DIC}}} \kappa\mathbb{E}\left[\Phi_\kappa(\boldsymbol{q}(\boldsymbol{\theta}),\boldsymbol{\theta})\right],\\ V_{\mathrm{BIC-IR}}(\kappa) &= \max_{\boldsymbol{q}\in\mathcal{Q}^{\mathrm{BIC}}} \kappa\,\mathbb{E}\left[\Phi_\kappa^{IR}(\boldsymbol{q},\boldsymbol{\theta})\right]\\ &= \max_{\boldsymbol{q}\in\mathcal{Q}^{\mathrm{DIC}}} \left\{\kappa\mathbb{E}\left[\Phi_\kappa(\boldsymbol{q}(\boldsymbol{\theta}),\boldsymbol{\theta})\right] + \sum_{i=1}^n (\kappa-w_i)\Gamma_i(\boldsymbol{q})\right\}. \end{aligned} \tag{5}$$

(iii) *If $\kappa > \max_{i\in\mathcal{N}} w_i$ and some fixed-$\kappa$ DIC–EIR optimal allocation rule $\boldsymbol{q}^*$ satisfies $\Gamma_i(\boldsymbol{q}^*)>0$ for some $i$, $V_{\mathrm{BIC-IR}}(\kappa) > V_{\mathrm{DIC-EIR}}(\kappa)$.*

The key to implementation is that higher types make allocations with larger own-type coefficients relatively more attractive in virtual-surplus terms. Holding opponents' reports fixed, two-sided regularity makes the virtual-surplus difference between an allocation with a larger coefficient and one with a smaller coefficient strictly increasing in own type. A pointwise maximizer therefore cannot select a smaller coefficient at a higher type: doing so would contradict optimality at one of the two types. Every measurable VSM selection is consequently DIC-implementable. The EIR worst-off normalization then supplies transfers that attain the pointwise virtual-surplus bound.

The value comparison follows by holding the allocation rule fixed and changing its participation normalization. Replacing EIR with IR reduces agent $i$'s minimum expected rent

by $\Gamma_i(\boldsymbol{q})$, raising the objective by $(\kappa - w_i)\Gamma_i(\boldsymbol{q})$. The BIC–IR value therefore adds these rent savings to each DIC rule's EIR value before reoptimizing, using the separate equivalences to restrict attention to DIC rules. A positive weighted rent gap at an EIR optimizer yields a strict gain even without changing the allocation. Appendix A.2.4 establishes attainment and the critical-type saddle-point representation, building on the IR approach of Loertscher and Wasser (2019) and Loertscher and Muir (2026) and extending it to EIR, where critical types may depend on opponents' reports.

Let $\boldsymbol{q}^{\text{eff}}(\boldsymbol{\theta})$ be the ex-post efficient allocation that maximizes $s(\boldsymbol{q}, \boldsymbol{\theta})$ for each $\boldsymbol{\theta}$.

**Remark 1** (Surplus-improving departures)**.** *If the VSM rule selects the status quo whenever it is a maximizer, then every departure from the status quo strictly increases total surplus:*

$$\boldsymbol{q}^{VSM}(\boldsymbol{\theta}) \neq \boldsymbol{q}^0 \quad \Longrightarrow \quad s(\boldsymbol{q}^{VSM}(\boldsymbol{\theta}), \boldsymbol{\theta}) > s(\boldsymbol{q}^0, \boldsymbol{\theta}).$$

*Proof.* Since $\Phi_\kappa(\boldsymbol{q}, \boldsymbol{\theta}) \leq s(\boldsymbol{q}, \boldsymbol{\theta})$ for every allocation and both equal zero at $\boldsymbol{q}^0$, the tie-breaking convention implies

$$\boldsymbol{q}^{VSM}(\boldsymbol{\theta}) \neq \boldsymbol{q}^0 \implies s(\boldsymbol{q}^{VSM}(\boldsymbol{\theta}), \boldsymbol{\theta}) \geq \Phi_\kappa(\boldsymbol{q}^{VSM}(\boldsymbol{\theta}), \boldsymbol{\theta}) > 0 = s(\boldsymbol{q}^0, \boldsymbol{\theta}).$$

□

Surplus-improving departures is familiar from optimal auctions and bilateral trade, where the optimal mechanism may forgo efficient trade but does not induce trade when no trade is efficient. With countervailing incentives, this property can fail under BIC–IR, as the partnership-dissolution illustration below shows. Remark 1 establishes that imposing DIC–EIR restores the property under the stated tie-breaking convention. Ex post participation prevents the mechanism from using gains at other opponents' type profiles to compensate an agent for losses from trade at the realized profile.

# 4 Pareto Improvement

We now ask when a DIC–EIR–EAND mechanism can be strictly improved upon for every agent and the designer. We construct improvements under DIC–IR that preserve EAND; the separate equivalences also permit implementation under BIC–EIR. All payoff comparisons are ex ante.

## 4.1 Strict Pareto Improvement

Strict Pareto improvement is possible if a DIC allocation rule raises expected surplus while preserving the interim own-type coefficients of agents whose rent gaps vanish under both the benchmark and the efficient rule.

**Theorem 3** (Strict Pareto improvement from interim participation)**.** *Maintain Assumption 1. Let* $(\boldsymbol{q}^\circ, \boldsymbol{t}^\circ)$ *be a DIC–EIR–EAND mechanism, and define*

$$Z := \{i \in \mathcal{N} : \Gamma_i(\boldsymbol{q}^\circ) + \Gamma_i(\boldsymbol{q}^{\text{eff}}) = 0\}.$$

*Suppose there exists* $\widehat{\boldsymbol{q}} \in \mathcal{Q}^{\text{DIC}}$ *satisfying*

$$S(\widehat{\boldsymbol{q}}) > S(\boldsymbol{q}^\circ), \tag{6}$$

$$M_i^{\widehat{\boldsymbol{q}}}(\theta_i) = M_i^{\boldsymbol{q}^\circ}(\theta_i) \qquad \text{for every } i \in Z \text{ and } \theta_i \in [0,1]. \tag{7}$$

*Then there exists a DIC–IR–EAND mechanism that gives every agent strictly higher ex ante expected utility and gives the designer strictly higher expected profit than* $(\boldsymbol{q}^\circ, \boldsymbol{t}^\circ)$.

The surplus condition implies that the benchmark mechanism is inefficient. Preserving interim own-type coefficients allows the agents in $Z$ to retain their interim utility functions under the new allocation rule; their allocations need not be preserved at every profile. The theorem applies even when the benchmark is Pareto efficient within DIC–EIR–EAND. It does not require an EAND implementation of $\widehat{\boldsymbol{q}}$, and the improving mechanism need not be efficient. The separate equivalences also permit a BIC–EIR implementation with the same expected payoffs.

The proof uses separation over DIC–IR mechanisms to obtain supporting weights if no strict improvement exists. Lemma 5 in Appendix A.3 then implies that every agent outside $Z$ has a positive benchmark rent gap. Reducing participation rents forces each such agent's supporting weight to equal the designer's weight. The supporting objective consequently depends only on total surplus and the utilities of agents in $Z$. Implementing $\widehat{\boldsymbol{q}}$ while preserving those utilities strictly raises this objective, a contradiction. A strict profit increase from the benchmark preserves EAND. Appendix A.3 gives the proof.

## 4.2 Conservation Conditions

A simple sufficient condition for strict Pareto improvement is stated here. A mechanism design problem satisfies the *conservation conditions* if

$$\sum_{i=1}^{n} m_i^k = 0, \qquad \sum_{i=1}^{n} a_i^k - C^k = 0 \qquad \text{for every } k \in \mathcal{K}. \tag{CC}$$

These conditions imply zero total surplus whenever all agents have the same type; individual intercepts and costs may nevertheless be nonzero. The partnership dissolution problem is an example. Recall that $m_i^k = \mathbb{1}\{i = k\} - r_i$ and $\sum_i r_i = 1$ so the own-type coefficients sum to zero under every $k$. In addition, $a_i^k = 0$ and $C^k = 0$ for all $k$, satisfying the second condition.

Together with countervailing incentives for every agent, these conditions ensure positive efficient-rule rent gaps. Thus $Z = \varnothing$, and for any inefficient benchmark the efficient rule satisfies the conditions of Theorem 3.

**Corollary 1** (Conservation and countervailing incentives)**.** *Maintain Assumption 1. Suppose* (CC) *holds and every agent faces countervailing incentives. Then* $\Gamma_i(\boldsymbol{q}^{\text{eff}}) > 0$ *for every* $i$*. Consequently, every DIC–EIR–EAND mechanism whose allocation rule is not ex post efficient on a set of positive probability admits a DIC–IR–EAND mechanism that strictly increases every agent's ex ante expected utility and the designer's expected profit.*

Under conservation, countervailing incentives rule out a common worst-off type under the efficient allocation rule. Fix an agent $i$ and let all opponents have type $t \in (0, 1)$. Under conservation, surplus from alternative $k$ is $(\theta_i - t)m_i^k$. If agent $i$ faces countervailing incentives, some alternatives have positive coefficients and others have negative coefficients. Efficient surplus is therefore zero at $\theta_i = t$ and strictly positive elsewhere. Since the agent's accumulated rent under efficiency differs from maximized surplus only by a constant, $t$ is his unique worst-off type at this opponents' profile. Varying $t$, and using continuity and full support to extend the comparison to positive-probability neighborhoods, rules out a common ex post worst-off type.

## 4.3 Limits of Strict Pareto Improvement

The following example shows that countervailing incentives for every agent are insufficient for strict Pareto improvement.

**Example 1** (Modified Hotelling auction)**.** *Two buyers have independent types* $\theta_i \sim U[0, 1]$ *and values* $1 - \theta_i$ *and* $\theta_i$ *for a unit at locations* 0 *and* 1*, respectively. A revenue-maximizing*

*seller can supply at most one unit in total, at either location, at zero cost. With $q_{i\ell}$ the probability that buyer $i$ receives the unit at location $\ell$, $a_i(\boldsymbol{q}) = q_{i0}$ and $m_i(\boldsymbol{q}) = q_{i1} - q_{i0}$.*

*Consider the following rule $\boldsymbol{q}^\circ$. If some buyer's type lies outside $[1/4, 3/4]$, the buyer farther from $1/2$ receives the unit at his preferred location; if both types lie in $[1/4, 3/4]$, each buyer receives the unit at each location with probability $1/4$.*

Both buyers face countervailing incentives, and $\boldsymbol{q}^\circ$ is inefficient on a set of positive probability. Its DIC–EIR–EAND implementation nevertheless yields expected revenue $17/24$, the maximum over all DIC–IR mechanisms. The seller therefore cannot strictly gain, ruling out a strict Pareto improvement for all parties.

Countervailing incentives coexist here with common worst-off types, so both the benchmark and efficient-rule rent gaps vanish. Under $\boldsymbol{q}^\circ$, a buyer with type in $[1/4, 3/4]$ either receives nothing or receives the symmetric lottery, so his own-type coefficient is zero; outside this interval it is nonpositive below and nonnegative above. Every type in $[1/4, 3/4]$ is therefore a common ex post worst-off type. Under efficiency, the buyer farther from $1/2$ receives the unit. A buyer of type $1/2$ therefore receives nothing for almost every opponent type; the exceptional all-$1/2$ profile may depend on tie-breaking. Type $1/2$ is a common ex post worst-off type. Theorem 1 gives $\Gamma_i(\boldsymbol{q}^\circ) = \Gamma_i(\boldsymbol{q}^{\text{eff}}) = 0$. Thus $Z = \mathcal{N}$. A DIC rule that increased surplus while preserving both buyers' interim coefficients would, with transfers preserving their interim utilities, increase the seller's revenue above the DIC–IR maximum. Hence no such rule satisfies (6)–(7). Online Appendix S1.1 verifies the allocation and rent-gap calculations.

# 5 Type Disclosure

We now consider a participant who chooses the mechanism to maximize his own expected utility. In partnership dissolution, for example, one partner selects the dissolution procedure while remaining a party to the resulting trade. Does this participant, agent 1, benefit from committing before learning his type rather than choosing the mechanism after his type becomes public?

We compare these two contracting regimes while the other agents' types remain private. Disclosure timing is taken as given; we do not model a decision to disclose or mechanism offers that signal private information. Section 5.1 specifies the incentive, participation, and budget constraints in each regime.

## 5.1 The Two Regimes

In the *ex ante regime*, agent 1 commits before learning his type; in the *transparent regime*, he chooses the mechanism after his type becomes public. The constraints differ as follows.

| | Ex ante choice | Choice after disclosure |
|---|---|---|
| Agent 1's information | Type unknown at commitment | Type $x$ public at choice |
| Agent 1's reporting | Cross-type DIC required | No private type to report |
| Agent 1's participation | EIR | None (status quo available) |
| Other agents' constraints | DIC and EIR | DIC and EIR |
| Budget constraint | Ex ante no deficit | No deficit conditional on every $x$ |

Disclosure removes agent 1's reporting constraint but requires the budget to satisfy no deficit conditional on each public type. Ex ante choice instead permits budget transfers across his types. After disclosure, agent 1 can choose the status quo, guaranteeing a nonnegative optimal conditional expected payoff without a separate participation constraint. Ex ante EIR requires his payoff to be nonnegative at every type profile.

Let $\mathcal{M}^{EA}$ denote the class of DIC–EIR–EAND mechanisms, and define $V^{EA}$ as the supremum of agent 1's expected payoff over this class:

$$\begin{aligned}\mathcal{M}^{EA} &:= \{(\boldsymbol{q},\boldsymbol{t}) : (\boldsymbol{q},\boldsymbol{t}) \text{ satisfies DIC, EIR, and EAND}\},\\ V^{EA} &:= \sup_{(\boldsymbol{q},\boldsymbol{t})\in\mathcal{M}^{EA}} \mathbb{E}[U_1(\boldsymbol{\theta})].\end{aligned}$$

Fix a public type $x \in [0,1]$; a transparent contract specifies allocations and transfers as functions of the other agents' reports. A contract is a pair $(\boldsymbol{q}_x, \boldsymbol{t}_x)$, where $\boldsymbol{q}_x : [0,1]^{n-1} \to \Delta(\mathcal{K})$, $t_{i,x} : [0,1]^{n-1} \to \mathbb{R}$, and $\boldsymbol{t}_x = (t_{i,x})_{i\in\mathcal{N}}$. Every reporting agent $i \neq 1$ must satisfy DIC and EIR, and the contract must satisfy conditional no deficit:

$$\mathbb{E}_{-1}\left[\sum_{i=1}^{n} t_{i,x}(\boldsymbol{\theta}_{-1}) - C(\boldsymbol{q}_x(\boldsymbol{\theta}_{-1}))\right] \geq 0.$$

Agent 1's payoff is $U_{1,x}(\boldsymbol{\theta}_{-1}) := a_1(\boldsymbol{q}_x(\boldsymbol{\theta}_{-1})) + x m_1(\boldsymbol{q}_x(\boldsymbol{\theta}_{-1})) - t_{1,x}(\boldsymbol{\theta}_{-1})$. Let $\mathcal{M}^T(x)$ denote the set of contracts satisfying these conditions, and define

$$v^T(x) := \sup_{(\boldsymbol{q}_x,\boldsymbol{t}_x)\in\mathcal{M}^T(x)} \mathbb{E}_{-1}[U_{1,x}(\boldsymbol{\theta}_{-1})].$$

We evaluate both regimes before agent 1's type is realized, so the ex ante value of trans-

parent choice is

$$V^T := \int_0^1 v^T(x)\, dF_1(x).$$

Restricting attention to mechanisms with expected budget balance leaves the value unchanged in both regimes. Any no-deficit slack can be rebated to agent 1 by lowering his transfer by a report-independent constant—globally in $EA$ and conditional on $x$ in $T$. The rebate leaves the allocation rule and every incentive constraint unchanged and, in $EA$, weakly relaxes agent 1's EIR constraint. Thus the suprema are unchanged when the budget balances in overall expectation in $EA$ and in conditional expectation for every $x$ in $T$. The following theorem establishes attainment.

## 5.2 The Equivalence

An ex ante optimal mechanism can remain optimal after every realization of agent 1's type, with the budget balancing in expectation conditional on each realization.

**Theorem 4** (Robust ex ante–transparent equivalence)**.** *Maintain Assumption 1. There exists a mechanism $(\boldsymbol{q}^*, \boldsymbol{t}^*) \in \mathcal{M}^{EA}$ attaining $V^{EA}$ such that, for every $x \in [0,1]$, its $x$-slice $(\boldsymbol{q}^*_x, \boldsymbol{t}^*_x) := (\boldsymbol{q}^*(x,\cdot), \boldsymbol{t}^*(x,\cdot))$ belongs to $\mathcal{M}^T(x)$, attains $v^T(x)$, and satisfies*

$$\mathbb{E}_{-1}\left[\sum_{i=1}^{n} t^*_{i,x}(\boldsymbol{\theta}_{-1}) - C(\boldsymbol{q}^*_x(\boldsymbol{\theta}_{-1}))\right] = 0.$$

*Consequently, $V^{EA} = V^T$.*

The theorem strengthens overall expected budget balance to conditional expected balance and supplies a common optimal mechanism for both regimes. The construction uses the same allocation rule and reporting agents' transfers before and after disclosure. Other transparent implementations may preserve agent 1's conditional expected payoff while changing his realized transfers and payoffs, as the posted-price implementation in Section 6.3 illustrates.

No deficit bounds agent 1's payoff by the fixed-$\kappa$ objective with $\kappa = w_1 = 1$ and $w_i = 0$ for $i \neq 1$. EAND gives

$$\begin{aligned}\mathbb{E}[U_1(\boldsymbol{\theta})] &\leq \mathbb{E}[U_1(\boldsymbol{\theta})] + \Pi(\boldsymbol{q}, \boldsymbol{t}) \\ &= \mathbb{E}\left[a_1(\boldsymbol{q}(\boldsymbol{\theta})) + \theta_1 m_1(\boldsymbol{q}(\boldsymbol{\theta})) + \sum_{i\neq 1} t_i(\boldsymbol{\theta}) - C(\boldsymbol{q}(\boldsymbol{\theta}))\right].\end{aligned}$$

The bound binds when the expected budget balances, and conditional no deficit gives the corresponding bound for each public type $x$, with expectation over $\boldsymbol{\theta}_{-1}$. Thus $\alpha_1 = 0$ and

$\alpha_i = 1$ for $i \neq 1$.

At these weights, agent 1 enters virtual surplus at his actual type, so the maximizing correspondence is independent of $F_1$. For each $i \neq 1$, write $\rho_i(\boldsymbol{q}, \theta_i) := a_i(\boldsymbol{q}) + \psi_i^B(\theta_i) m_i^+(\boldsymbol{q}) - \psi_i^S(\theta_i) m_i^-(\boldsymbol{q})$, where the virtual value and virtual cost are evaluated at $\alpha_i = 1$. Thus the allocation objective is

$$\Phi^1(\boldsymbol{q}, x, \boldsymbol{\theta}_{-1}) := a_1(\boldsymbol{q}) + x m_1(\boldsymbol{q}) + \sum_{i \neq 1} \rho_i(\boldsymbol{q}, \theta_i) - C(\boldsymbol{q}).$$

This is $\Phi_1$ at the stated weights. Choose, independently of $F_1$, a measurable pointwise maximizer

$$\boldsymbol{q}^1(x, \boldsymbol{\theta}_{-1}) \in \arg\max_{\boldsymbol{q} \in \Delta(\mathcal{K})} \Phi^1(\boldsymbol{q}, x, \boldsymbol{\theta}_{-1}),$$

and let $V^1(x; \boldsymbol{\theta}_{-1}) := \max_{\boldsymbol{q} \in \Delta(\mathcal{K})} \Phi^1(\boldsymbol{q}, x, \boldsymbol{\theta}_{-1})$.

The common bound is attained by giving the reporting agents their EIR-normalized VSM transfers and charging agent 1 the allocation cost net of their virtual contributions:

$$t_1^*(x, \boldsymbol{\theta}_{-1}) := C(\boldsymbol{q}^1(x, \boldsymbol{\theta}_{-1})) - \sum_{i \neq 1} \rho_i(\boldsymbol{q}^1(x, \boldsymbol{\theta}_{-1}), \theta_i).$$

The reporting agents' VSM transfers ensure DIC and EIR. With the displayed transfer, agent 1's payoff from any report equals virtual surplus evaluated at his true type and the allocation selected by that report. Truthful reporting selects a maximizer, and the availability of the status quo makes his truthful payoff nonnegative. The construction permits the reporting agents' worst-off types to vary across opponents' reports.

For each public type $x$, the reporting agents' expected transfers equal their expected virtual contributions, so the constructed mechanism balances the budget in conditional expectation. No expected budget subsidies across agent 1's types are needed to attain the ex ante optimum, and removing his reporting constraints after disclosure cannot raise the common bound. The same mechanism therefore attains

$$v^T(x) = \mathbb{E}_{-1}[V^1(x; \boldsymbol{\theta}_{-1})], \qquad V^{EA} = \mathbb{E}[V^1(\theta_1; \boldsymbol{\theta}_{-1})].$$

Appendix A.4 proves the payoff bounds and verifies incentive compatibility, participation, and conditional budget balance.

Mylovanov and Tröger (2014) study the comparison between ex ante and transparent contracting in a broader analysis of informed-principal mechanism selection. In linear private-value environments, they give a monotonicity condition under which the principal's private

information has no value and show that the comparison can change when payoffs are non-monotonic in type. Our result establishes equivalence under DIC and EIR even with countervailing incentives; it compares the two contracting regimes rather than characterizing equilibrium mechanism offers by an informed principal.

# 6 Partnership Dissolution

We specialize the partnership-dissolution setup of Section 2.1 to two partners with independent types drawn from $F_1$ and $F_2$, maintaining Assumption 1. Initial shares satisfy $r_1, r_2 \in (0,1)$ and $r_1 + r_2 = 1$; boundary ownership is considered separately. The example quantifies optimal trade, rent saving, and the value of commitment before type disclosure. We call DIC–EIR mechanisms *robust* because their guarantees hold at every type profile, although optimal design can depend on the type distributions. Online Appendix S1.2 provides the calculations and implementation details.

## 6.1 From virtual bids and asks to trading rules

In partnership dissolution, an agent's own-type coefficient is his net acquisition, $q_i - r_i$. A partner is a net buyer when his final share exceeds his initial ownership and a net seller when it falls below it. Both DIC–EIR and BIC–IR evaluate these acquisition incentives using virtual values and virtual costs. The distinction lies in whether participation rents are normalized separately at each opponent's type or after averaging over that type.

**Robust optimal allocation.** Under DIC–EIR, an optimal mechanism solves the pointwise VSM problem by Theorem 2: for each $\boldsymbol{\theta}$,

$$\max_{q_1(\boldsymbol{\theta})\in[0,1]} (\psi_1^B(\theta_1) - \psi_2^S(\theta_2))(q_1(\boldsymbol{\theta}) - r_1)^+ + (\psi_2^B(\theta_2) - \psi_1^S(\theta_1))(r_1 - q_1(\boldsymbol{\theta}))^+.$$

Subject to two-sided regualarity assumption, the VSM allocation rule is

$$q_1^{VSM}(\boldsymbol{\theta}) = \begin{cases} 1, & \psi_1^B(\theta_1) > \psi_2^S(\theta_2), \\ 0, & \psi_2^B(\theta_2) > \psi_1^S(\theta_1), \\ r_1, & \text{otherwise.} \end{cases}$$

Since $\psi_i^B(\theta_i) \le \theta_i \le \psi_i^S(\theta_i)$, the two strict inequalities cannot hold simultaneously. The mechanism therefore transfers the entire asset to a partner whenever his virtual value exceeds the other partner's virtual cost, and it retains the initial shares otherwise. For each

opponent's report, any such type is an ex post worst-off type: under the EIR-normalized transfers, the partner retains his initial share and makes zero transfers.

**Interim optimal allocation.** Under BIC–IR, optimal allocations compare ironed virtual types constructed around each partner's interim worst-off type.[8] Types in an ironing interval receive their initial share in expectation, but may buy at some opponent types and sell at others. Ironing therefore need not produce a no-trade region: retaining the initial shares at a profile is one possible allocation at equal ironed virtual types, subject to the interim worst-off conditions (i.e., no trade in expectation). Because these conditions require expected allocation to equal the initial ownership $r_i$, they can affect the ironed virtual types and the trading regions.

Initial ownership enters the two problems differently. At fixed welfare and profit weights, positive initial shares scale the DIC–EIR gains from trade without changing their signs, so ownership affects the quantities traded but not the trading regions. The fixed-weight qualification matters: when EAND is imposed, its supporting profit weight may itself depend on ownership. The examples below show how these differences affect optimal allocations and payoffs.

## 6.2 Trading regions and optimal values

Set $\theta_1, \theta_2 \sim U[0,1]$ independently and $r_1 = r_2 = 1/2$. We compare DIC–EIR and BIC–IR optima for three objectives: expected profit, the partners' average utility, and partner 1's utility. The BIC–IR benchmarks correspond to the profit-maximizing mechanism of Loertscher and Wasser (2019), the efficient dissolution mechanism of Cramton et al. (1987), and the partner-optimal mechanism of Mylovanov and Tröger (2014). Both problems impose EAND. Its shadow value can differ across constraint classes, so the supporting profit weight $\kappa$ need not be the same in the two problems.

Figure 1 illustrates how ex post and interim participation constraints shape optimal trading regions. Across all three objectives under BIC–IR, a partner with type 1/2 has zero expected net trade, although she may buy from some opponent types and sell to others. Under DIC–EIR, the white regions instead retain initial ownership with zero transfers.

[8] For the BIC–IR optimal mechanism construction, we follow Loertscher and Marx (2026, Theorem 1 and Proposition 1). Their multiplier $\rho$ corresponds to our $\kappa$, and their virtual-type weight $w_i/\rho$ to $1-\alpha_i = w_i/\kappa$.

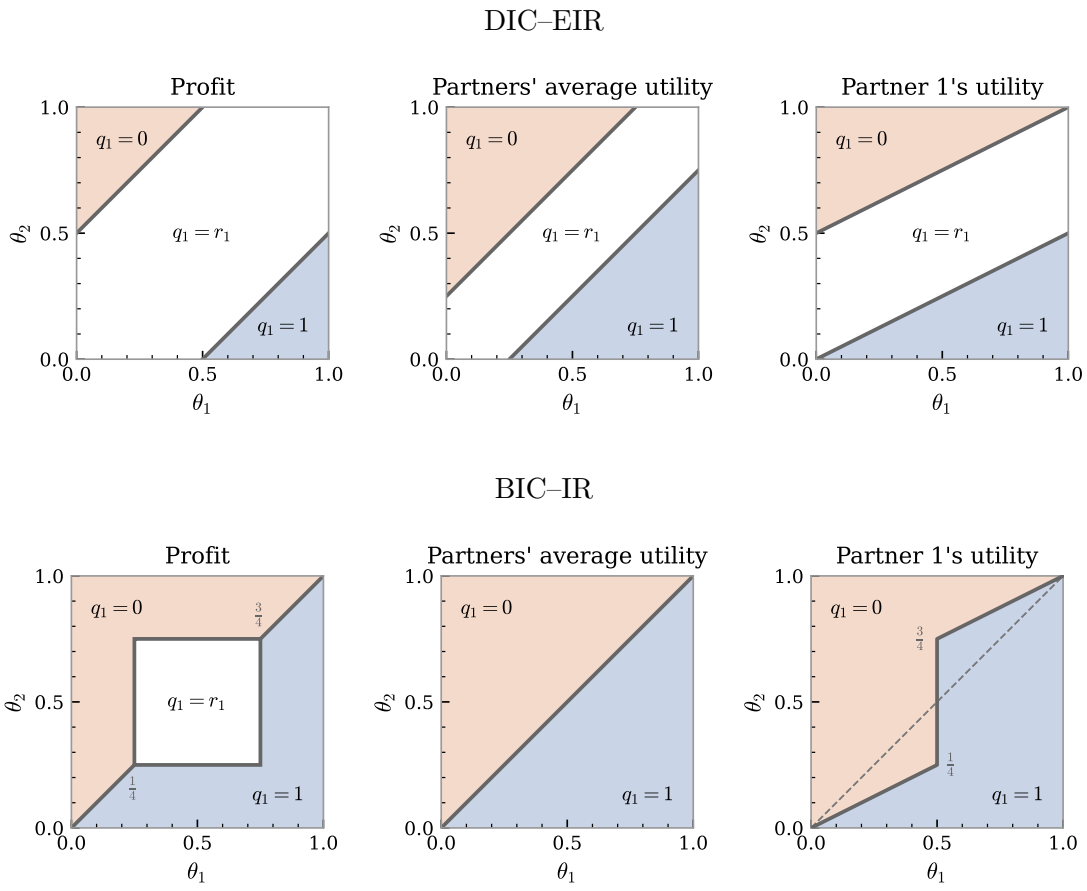



Figure 1: Optimal allocations under EAND with independent uniform types and equal initial shares: DIC–EIR (upper row) and BIC–IR (lower row). In the lower-left panel, retaining initial shares on the central square is an optimal tie-breaking choice.

## 6.3 Transparency and the commitment premium

Continue with independent uniform types and return to arbitrary interior ownership. Theorem 4 gives the same optimal expected payoff before and after disclosure under the robust constraints. Write $x = \theta_1$ and $y = \theta_2$. We apply the two regimes defined in Section 5.1.

After disclosure, partner 1 offers a per-unit bid $b$ and ask $a$, with $0 \leq b \leq a \leq 1$. Partner 2 sells below the bid, buys above the ask, and otherwise retains his share. Partner 1's conditional expected payoff is

$$u_1(b, a \mid x) = r_2 b(x - b) + r_1(1 - a)(a - x).$$

Maximizing the two terms gives $b(x) = x/2$ and $a(x) = (x+1)/2$, the transparent benchmark of Mylovanov and Tröger (2014, Proposition 10). Remitting the buyer's payment to the seller balances the budget at every profile. The virtual-surplus bound in Section 5 establishes optimality over all admissible contracts.

Ex ante implementation uses the same allocation but different payments for partner 1. Prices tied to his private report would let him manipulate his trading price. Instead, critical payments determined by partner 2's report give him realized utility

$$U_1^D(x, y) = r_2(x - 2y)^+ + r_1(2y - 1 - x)^+,$$

where $z^+ := \max\{z, 0\}$. Truthful reporting selects his best trading outcome, while partner 2 faces the posted bid and ask. The online appendix verifies DIC, EIR, and conditional expected budget balance. Integrating over $y$ gives

$$\mathbb{E}_y[U_1^D(x, y)] = \frac{r_2 x^2}{4} + \frac{r_1(1 - x)^2}{4} = \mathbb{E}_y[U_1^T(x, y)],$$

where $T$ denotes the transparent implementation. The equality concerns conditional expected payoff; realized payoffs can differ. Unlike the posted-price contract, the ex ante implementation need not balance the budget at every profile.

Averaging over $x$ gives the optimal ex ante and transparent values

$$V^{EA} = V^T = \frac{r_1 + r_2}{12} = \frac{1}{12},$$

as shown by Theorem 4

# 7 Conclusion

The rent gap measures the additional expected participation rent required by EIR at a fixed DIC allocation rule. These costs shape optimal robust mechanisms and the gains from relaxing participation. Under the maintained regularity assumptions, a surplus-increasing DIC allocation rule that preserves interim coefficients for agents with zero benchmark and efficient-rule rent gaps ensures that a DIC–EIR–EAND benchmark admits an improvement that strictly raises every agent's ex ante expected utility and the designer's expected profit while preserving DIC, IR, and EAND. Such improvements do not require first-best implementation to be feasible.

When a participant chooses the mechanism, a common optimum attains both the ex ante and transparent values, balancing the budget in expectation conditional on every realization of that participant's type. Commitment before learning his type therefore has no value under the robust constraints. The partnership application illustrates this equivalence and contrasts it with a positive BIC–IR commitment premium under interior ownership, showing how the value of commitment depends on the incentive and participation guarantees.

# A Proofs

## A.1 Proofs for Section 2

*Proof of Lemma 1.* First compare BIC–IR and DIC–IR. The BIC–DIC equivalence theorem of Gershkov et al. (2013) yields the result when all own-type coefficients are nonnegative. To apply it here, choose $b_i > -\min_k m_i^k$ for each $i$ and replace $m_i^k$ by $m_i^k + b_i$. This adds the allocation- and report-independent term $b_i\theta_i$ to agent $i$'s payoff and leaves all incentive comparisons unchanged. Apply the theorem in the shifted environment and then undo the shift. Since the transformation preserves interim utilities in the shifted environment, subtracting the same term $b_i\theta_i$ restores each type's original interim utility, and hence IR, in the original environment.

The construction also preserves expected gross surplus and the ex ante probability of every alternative. These are distinct preservation properties: the former concerns the joint distribution of types and allocations, whereas the latter preserves expected allocation cost. Undoing the coefficient shift subtracts the same expected term $\sum_i b_i \mathbb{E}[\theta_i]$ from gross surplus under both mechanisms. Preservation of expected gross surplus and interim utilities implies preservation of total expected transfers; together with unchanged expected cost, this preserves expected profit, $W_\kappa$, EAND, and $J$. The reverse inequalities follow because DIC implies BIC.

Next compare BIC–IR and BIC–EIR. Fix a BIC–IR mechanism $(\boldsymbol{q}, \boldsymbol{t})$. Keeping the allocation rule fixed, define

$$t_i^E(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) := a_i(\boldsymbol{q}(\hat{\theta}_i, \boldsymbol{\theta}_{-i})) + \hat{\theta}_i m_i(\boldsymbol{q}(\hat{\theta}_i, \boldsymbol{\theta}_{-i})) - \mathbb{E}_{-i}\left[U_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\right].$$

Under $(\boldsymbol{q}, \boldsymbol{t}^E)$, a truthful type's ex post payoff is $\mathbb{E}_{-i}[U_i(\theta_i, \boldsymbol{\theta}_{-i})] \geq 0$, so the new mechanism satisfies EIR. To verify BIC, use independence to evaluate the new interim deviation payoff. For every true type $\theta_i$ and report $\hat{\theta}_i$, the new interim deviation payoff is

$$\begin{aligned}&\mathbb{E}_{-i}\left[U_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\right] + (\theta_i - \hat{\theta}_i)\mathbb{E}_{-i}\left[m_i(\boldsymbol{q}(\hat{\theta}_i, \boldsymbol{\theta}_{-i}))\right] \\ &\quad = \mathbb{E}_{-i}\left[U_i(\theta_i, \hat{\theta}_i, \boldsymbol{\theta}_{-i})\right].\end{aligned}$$

Thus the construction preserves BIC; it need not preserve DIC. Iterated expectations give $\mathbb{E}[t_i^E] = \mathbb{E}[t_i]$. The allocation rule, interim utilities, and expected transfers are unchanged, so the construction preserves EAND, $J$, and $W_\kappa$. Because EIR implies IR, the reverse inequalities also hold.[9] □

## A.2 Proofs for Section 3

We first establish the primitive characterization and the fixed-rule participation values. We then construct the optimal DIC–EIR mechanism, establish critical-type saddle points and attainment, complete the value comparison, and derive the supporting-weight result under EAND.

### A.2.1 Primitive Characterization of Countervailing Incentives

*Proof of Lemma 2.* Fix an agent $i$. We compare the universal ex post worst-off set $\Omega_i^{EP}$ with the intersection $\bigcap_{\boldsymbol{q} \in \mathcal{Q}^{\mathrm{BIC}}} \Omega_i(\boldsymbol{q})$ of interim worst-off sets. If $m_i^k \geq 0$ for every $k$, then accumulated rent and its expectation are weakly increasing in own type under every feasible allocation rule. Type 0 therefore belongs to both sets in the asserted equality. If $m_i^k > 0$ for some $k$, the constant allocation rule selecting that alternative makes type 0 the unique worst-off type both interim and at every opponents' profile. This rule is DIC-implementable for every agent, and hence BIC-implementable. Both sets therefore equal $\{0\}$. The uniqueness conclusion concerns types that are worst off under every allocation rule; an individual allocation rule can have several worst-off types.

[9] Remark 6 of Chen et al. (2019) establishes the participation equivalence in a more general environment by combining their nonconstructive purification result with an explicit transfer adjustment.

If $m_i^k \leq 0$ for every $k$, with strict inequality for some $k$, the mirror argument gives $\{1\}$ for both sets.

If $m_i^k = 0$ for every $k$, all types have the same accumulated rent under every allocation rule, so both sets equal $[0,1]$.

Finally, suppose $m_i^{k^-} < 0 < m_i^{k^+}$ for some alternatives $k^-$ and $k^+$. The constant allocation rule selecting $k^-$ has type 1 as its unique worst-off type; the constant allocation rule selecting $k^+$ has type 0 as its unique worst-off type. Both rules are DIC-implementable for every agent and hence BIC-implementable. Neither set can contain a type, so both sets are empty. These cases exhaust all possible coefficient signs and establish the equality and the primitive characterization. □

### A.2.2 Participation Normalizations and Fixed-Rule Values

For $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}$, define the profilewise worst-off set by

$$\Omega_i^{EP}(\boldsymbol{q}, \boldsymbol{\theta}_{-i}) := \arg\min_{x \in [0,1]} R_i^{\boldsymbol{q}}(x, \boldsymbol{\theta}_{-i}).$$

Accumulated rent is jointly measurable, continuous and convex in own type, and uniformly bounded. Its profilewise minimum is therefore measurable, and the measurable maximum theorem supplies a measurable minimizing selection. Its interim expectation is continuous and attains its minimum on $[0,1]$. The IR and EIR worst-off normalizations in (1) consequently define measurable, integrable utilities and transfers. The envelope formula shows that these normalizations attain minimum expected participation rents; any other feasible normalization adds a nonnegative interim constant under IR or nonnegative profilewise constants under EIR.

**Proposition 1** (Fixed-rule virtual surplus and rent gaps)**.** *Fix $\kappa > 0$ with $\kappa \geq \max_i w_i$. For every $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{BIC}}$,*

$$\max_{\boldsymbol{t}:(\boldsymbol{q},\boldsymbol{t})\ \mathrm{BIC\text{–}IR}} W_\kappa(\boldsymbol{q}, \boldsymbol{t}) = \kappa \mathbb{E}[\Phi_\kappa^{IR}(\boldsymbol{q}, \boldsymbol{\theta})]. \tag{8}$$

*The maximum is attained by normalizing minimum interim utility to zero. If $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}$, the same value is attained with DIC–IR transfers, and*

$$\begin{aligned} \max_{\boldsymbol{t}:(\boldsymbol{q},\boldsymbol{t})\ \mathrm{DIC\text{–}EIR}} W_\kappa(\boldsymbol{q}, \boldsymbol{t}) &= \kappa \mathbb{E}[\Phi_\kappa(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{\theta})], \\ \kappa \mathbb{E}[\Phi_\kappa^{IR}(\boldsymbol{q}, \boldsymbol{\theta})] &= \kappa \mathbb{E}[\Phi_\kappa(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{\theta})] + \sum_i (\kappa - w_i) \Gamma_i(\boldsymbol{q}). \end{aligned} \tag{9}$$

*The EIR maximum is attained by the profilewise worst-off normalization. These conclusions*

*do not require two-sided regularity.*

*Proof of Proposition 1.* Fix $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{BIC}}$. BIC makes $M_i^{\boldsymbol{q}}$ weakly increasing. Its negative values precede its positive values, so the interim envelope with minimum utility zero can be written as

$$u_i^{\boldsymbol{q}}(x) = \int_0^x \big(M_i^{\boldsymbol{q}}(s)\big)^+ \, ds + \int_x^1 \big(M_i^{\boldsymbol{q}}(s)\big)^- \, ds.$$

This expression has derivative $M_i^{\boldsymbol{q}}$ almost everywhere, is nonnegative, and equals zero at an interim worst-off type. It also covers boundary worst-off types and intervals of zero coefficients. Every BIC–IR implementation of the fixed rule has interim utility $u_i^{\boldsymbol{q}}(x) + b_i$ for some $b_i \geq 0$. Since $\kappa - w_i \geq 0$, setting $b_i = 0$ maximizes the objective over transfers.

Bounded coefficients permit Fubini's theorem, which gives

$$\mathbb{E}_i[u_i^{\boldsymbol{q}}(\theta_i)] = \int_0^1 \Big[(1 - F_i(x))\big(M_i^{\boldsymbol{q}}(x)\big)^+ + F_i(x)\big(M_i^{\boldsymbol{q}}(x)\big)^-\Big] \, dx. \tag{10}$$

Moreover, iterated expectations give $\mathbb{E}[\theta_i m_i(\boldsymbol{q}(\boldsymbol{\theta}))] = \mathbb{E}_i[\theta_i M_i^{\boldsymbol{q}}(\theta_i)]$. Substituting these identities into $W_\kappa = \kappa S - \sum_i (\kappa - w_i)\mathbb{E}[U_i]$, and using the definitions of $\psi_i^B$ and $\psi_i^S$, proves (8). The density assumption makes the virtual types well defined and integrable.

For attainment, set

$$t_i(\boldsymbol{\theta}) = a_i(\boldsymbol{q}(\boldsymbol{\theta})) + \theta_i m_i(\boldsymbol{q}(\boldsymbol{\theta})) - u_i^{\boldsymbol{q}}(\theta_i).$$

The interim envelope and monotonicity of $M_i^{\boldsymbol{q}}$ imply BIC, and the normalization implies IR. If $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}$, instead use the truthful payoff

$$U_i(\boldsymbol{\theta}) = R_i^{\boldsymbol{q}}(\theta_i, \boldsymbol{\theta}_{-i}) - \min_{x \in [0,1]} \mathbb{E}_{-i}[R_i^{\boldsymbol{q}}(x, \boldsymbol{\theta}_{-i})]$$

and set transfers equal to gross payoff minus $U_i$. The subtracted term is independent of reports. Pointwise monotonicity and the envelope formula imply DIC, while expected utility conditional on own type equals $u_i^{\boldsymbol{q}}$, implying IR. Thus the fixed-rule value is the same under DIC–IR.

For EIR, apply the same envelope calculation separately at each opponents' profile. DIC makes the coefficient schedule nondecreasing, so the minimum-rent payoff is

$$U_i^{EIR}(\boldsymbol{\theta}) = \int_0^{\theta_i} m_i^+(\boldsymbol{q}(s, \boldsymbol{\theta}_{-i})) \, ds + \int_{\theta_i}^1 m_i^-(\boldsymbol{q}(s, \boldsymbol{\theta}_{-i})) \, ds.$$

Fubini's theorem gives its expected rent by replacing $M_i^{\boldsymbol{q}}$ in (10) with the profilewise coeffi-

cient and then averaging over opponents' types. Substitution into the surplus identity gives the first equality in (9). The envelope and pointwise monotonicity give DIC, and the nonnegative normalized payoff gives EIR. Finally, Theorem 1 identifies the difference between minimum EIR and IR expected rents as $\Gamma_i(\boldsymbol{q})$. Subtracting the two fixed-rule objectives proves the second equality. □

### A.2.3 Pointwise Maximization and DIC–EIR Optimality

We now impose two-sided regularity and construct an optimal DIC–EIR mechanism directly from the fixed-rule value formula.

**Lemma 3** (Measurability and monotonicity)**.** *Under Assumption 1, there is a measurable VSM allocation rule. Every such selection has a weakly increasing own-type coefficient schedule: for every $i$ and every $\boldsymbol{\theta}_{-i}$, $m_i(\boldsymbol{q}^{VSM}(\theta_i, \boldsymbol{\theta}_{-i}))$ is weakly increasing in $\theta_i$.*

*Proof. Measurable selection.* Since $\Delta(\mathcal{K})$ is compact, $\Phi_\kappa(\cdot, \boldsymbol{\theta})$ is continuous for every $\boldsymbol{\theta}$, and $\Phi_\kappa(\boldsymbol{q}, \cdot)$ is measurable for every $\boldsymbol{q}$, the measurable maximum theorem (Aliprantis and Border, 2006, Theorem 18.19) implies that the pointwise argmax correspondence is nonempty and compact-valued and admits a measurable selection. Fix one such selection and denote it by $\boldsymbol{q}^{VSM}$.

*Monotonicity.* Fix $i$, $\boldsymbol{\theta}_{-i}$, and $\theta_i' < \theta_i''$, and let $\boldsymbol{q}'$ and $\boldsymbol{q}''$ be the selected allocations at these two type profiles. Suppose $m_i(\boldsymbol{q}') > m_i(\boldsymbol{q}'')$. Holding $\boldsymbol{q}'$, $\boldsymbol{q}''$, and $\boldsymbol{\theta}_{-i}$ fixed, every term other than agent $i$'s virtual-type term is constant in $\theta_i$; the difference between agent $i$'s terms is a positive multiple of $\psi_i^B$ if both coefficients are nonnegative, a positive multiple of $\psi_i^S$ if both are nonpositive, and a positive linear combination of $\psi_i^B$ and $\psi_i^S$ if they have opposite signs. Hence $\Delta(\theta_i) := \Phi_\kappa(\boldsymbol{q}', \theta_i, \boldsymbol{\theta}_{-i}) - \Phi_\kappa(\boldsymbol{q}'', \theta_i, \boldsymbol{\theta}_{-i})$ is strictly increasing. Optimality gives $\Delta(\theta_i') \geq 0$ and $\Delta(\theta_i'') \leq 0$, contradicting $\theta_i' < \theta_i''$. Therefore, $m_i(\boldsymbol{q}^{VSM}(\theta_i', \boldsymbol{\theta}_{-i})) \leq m_i(\boldsymbol{q}^{VSM}(\theta_i'', \boldsymbol{\theta}_{-i}))$. □

*Proof of Theorem 2(i).* For every DIC–EIR mechanism, Proposition 1 gives

$$W_\kappa(\boldsymbol{q}, \boldsymbol{t}) \leq \kappa \mathbb{E}[\Phi_\kappa(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{\theta})] \leq \kappa \mathbb{E}\left[\max_{\boldsymbol{x} \in \Delta(\mathcal{K})} \Phi_\kappa(\boldsymbol{x}, \boldsymbol{\theta})\right].$$

Lemma 3 supplies a measurable pointwise maximizer that is DIC-implementable. The EIR worst-off normalization attains its fixed-rule value by Proposition 1, so the VSM mechanism attains both bounds and is optimal. The normalization yields DIC and EIR at every profile and measurable, integrable transfers as established in Section A.2.2.

Conversely, an optimal mechanism must attain both bounds. Its shortfall from the pointwise maximum is nonnegative and has expectation zero, so its allocation maximizes $\Phi_\kappa$ for

$F$-almost every profile. This proves necessity as well as sufficiency, independently of the saddle-point argument below. □

### A.2.4 Critical-Type Representations and Saddle Points

Fix $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}$ and a measurable critical-type rule $\boldsymbol{c}$, with $c_i(\boldsymbol{\theta}_{-i}) \in [0,1]$ almost everywhere. Normalize truthful payoff by

$$U_i(\boldsymbol{\theta}) = R_i^{\boldsymbol{q}}(\theta_i, \boldsymbol{\theta}_{-i}) - R_i^{\boldsymbol{q}}(c_i(\boldsymbol{\theta}_{-i}), \boldsymbol{\theta}_{-i}).$$

Substituting this payoff into the designer's objective defines

$$\widetilde{W}_\kappa(\boldsymbol{q}, \boldsymbol{c}) := \kappa S(\boldsymbol{q}) - \sum_{i=1}^{n} (\kappa - w_i) \mathbb{E}\left[R_i^{\boldsymbol{q}}(\theta_i, \boldsymbol{\theta}_{-i}) - R_i^{\boldsymbol{q}}(c_i(\boldsymbol{\theta}_{-i}), \boldsymbol{\theta}_{-i})\right]. \tag{11}$$

At a fixed allocation rule, the critical-type rule affects the objective only through the rent normalization. An arbitrary critical-type normalization need not satisfy IR or EIR; the corresponding participation constraint is satisfied when interim or profilewise worst-off types are selected, respectively.

For a critical type $x \in [0,1]$, the corresponding *virtual type* is

$$\psi_i(\theta_i; x) := \begin{cases} \psi_i^S(\theta_i), & \theta_i \le x, \\ \psi_i^B(\theta_i), & \theta_i > x. \end{cases} \tag{12}$$

It equals virtual cost below the critical type and virtual value above it. Because types are independent and atomless, the value assigned at $\theta_i = c_i(\boldsymbol{\theta}_{-i})$ does not affect any expectation.

For fixed opponents' reports, Fubini's theorem gives

$$\mathbb{E}_i[U_i \mid \boldsymbol{\theta}_{-i}] = \int_{c_i(\boldsymbol{\theta}_{-i})}^{1} [1 - F_i(s)] m_i(\boldsymbol{q}(s, \boldsymbol{\theta}_{-i}))\, ds - \int_0^{c_i(\boldsymbol{\theta}_{-i})} F_i(s) m_i(\boldsymbol{q}(s, \boldsymbol{\theta}_{-i}))\, ds.$$

Substituting this expression into the objective gives the equivalent virtual-surplus representation

$$\widetilde{W}_\kappa(\boldsymbol{q}, \boldsymbol{c}) = \kappa \mathbb{E}\left[\sum_{i=1}^{n} \{a_i(\boldsymbol{q}(\boldsymbol{\theta})) + m_i(\boldsymbol{q}(\boldsymbol{\theta})) \psi_i(\theta_i; c_i(\boldsymbol{\theta}_{-i}))\} - C(\boldsymbol{q}(\boldsymbol{\theta}))\right].$$

We call $\widetilde{W}_\kappa$ the *critical-type virtual surplus*.

Under IR, critical types are vectors $\boldsymbol{x} \in [0,1]^n$, identified with the constant rules $c_i(\boldsymbol{\theta}_{-i}) = x_i$. Under EIR, $\mathcal{C}^{\mathrm{EIR}}$ denotes the class of measurable rules $\boldsymbol{\omega}$ satisfying $\omega_i(\boldsymbol{\theta}_{-i}) \in [0,1]$ almost everywhere. We write $\widetilde{W}_\kappa(\boldsymbol{q}, \boldsymbol{x})$ for the objective evaluated at the corresponding constant

rule.

The IR saddle-point approach and existence results have precedents in Loertscher and Wasser (2019) for partnership dissolution and Loertscher and Muir (2026) for Hotelling auctions. The proposition below establishes the IR representation and existence result in our finite social-choice environment and also treats EIR, under which critical types may depend on opponents' reports.

**Proposition 2** (Saddle-point representations and existence)**.** *Maintain Assumption 1. Fix $\kappa > 0$ with $\alpha_i \geq 0$ for every $i$. Then*

$$\begin{aligned} V_{\mathrm{BIC-IR}}(\kappa) = V_{\mathrm{DIC-IR}}(\kappa) &= \sup_{\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}} \min_{\boldsymbol{x} \in [0,1]^n} \widetilde{W}_\kappa(\boldsymbol{q}, \boldsymbol{x}), \\ V_{\mathrm{DIC-EIR}}(\kappa) &= \sup_{\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}} \min_{\boldsymbol{\omega} \in \mathcal{C}^{\mathrm{EIR}}} \widetilde{W}_\kappa(\boldsymbol{q}, \boldsymbol{\omega}). \end{aligned} \tag{13}$$

*Each outer supremum is attained, and both max–min problems admit a saddle point. Every primal optimizer pairs with every dual optimizer to form a saddle point.*

*Proof. Fixed-rule critical-type minima.* For fixed $\boldsymbol{q}$, equation (11) shows that dependence on critical types is, up to a constant,

$$\sum_i (\kappa - w_i) \mathbb{E}_{-i}[R_i^{\boldsymbol{q}}(c_i(\boldsymbol{\theta}_{-i}), \boldsymbol{\theta}_{-i})].$$

The nonnegative weights make interim worst-off types minimizing choices under IR, and profilewise worst-off types minimizing choices under EIR. Such choices exist and are measurable where required by Section A.2.2. If a weight is zero, any critical type minimizes that agent's term. Proposition 1 therefore yields

$$\begin{aligned} \min_{\boldsymbol{\omega} \in \mathcal{C}^{\mathrm{EIR}}} \widetilde{W}_\kappa(\boldsymbol{q}, \boldsymbol{\omega}) &= \kappa \mathbb{E}[\Phi_\kappa(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{\theta})], \\ \min_{\boldsymbol{x} \in [0,1]^n} \widetilde{W}_\kappa(\boldsymbol{q}, \boldsymbol{x}) &= \kappa \mathbb{E}[\Phi_\kappa^{IR}(\boldsymbol{q}, \boldsymbol{\theta})] \\ &= \kappa \mathbb{E}[\Phi_\kappa(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{\theta})] + \sum_i (\kappa - w_i) \Gamma_i(\boldsymbol{q}). \end{aligned} \tag{14}$$

Taking suprema over DIC rules and applying Lemma 1 gives (13). It remains to establish attainment and saddle points.

*Compactness and minimax in the relaxed space.*

First work with the relaxed set of allocation rules $\widehat{\mathcal{Q}} \subset L^2(F)^K$: its elements satisfy the simplex constraints almost everywhere, and each own-type coefficient schedule admits a weakly increasing representative for $F_{-i}$-almost every opponents' profile. Elements of $L^2$

are equivalence classes modulo $F$-null sets. Every allocation rule in $\mathcal{Q}^{\mathrm{DIC}}$ defines an element of $\widehat{\mathcal{Q}}$, but compactness of this relaxed set alone does not establish implementation at every report profile. We recover pointwise DIC mechanisms after the minimax argument. The lower bound on $f_i$ implies that $\psi_i^S$ and $\psi_i^B$ are bounded and hence belong to $L^2(F_i)$.

The almost-everywhere simplex constraints define a convex, norm-closed, and bounded subset of $L^2(F)^K$. For each $i$, the map from an allocation rule to its own-type coefficient is continuous and linear. The subset of $L^2(F)$ whose $\theta_i$-sections admit weakly increasing representatives for $F_{-i}$-almost every $\boldsymbol{\theta}_{-i}$ is convex and norm closed. To see closedness, take an $L^2$-convergent sequence and use Aliprantis and Border (2006, Theorem 13.6) to pass to an almost-everywhere convergent subsequence. Fubini's theorem then implies convergence for almost every $\theta_i$ in almost every opponents' profile section. On the full-measure set of convergence in each such section, the limit preserves the order inequalities of the approximating monotone functions and therefore admits a weakly increasing extension. The DIC restrictions are thus convex and norm closed. Because norm-closed convex sets are weakly closed (Aliprantis and Border, 2006, Theorem 5.98), while $L^2$ is reflexive (Aliprantis and Border, 2006, Corollary 13.27) and its closed unit ball is weakly compact (Aliprantis and Border, 2006, Theorem 6.25), $\widehat{\mathcal{Q}}$ is weakly compact and convex. It is nonempty because it contains the constant status-quo rule.

For the minimax argument, identify measurable EIR critical-type rules that agree almost everywhere and use their quotient representation in the product of the corresponding $L^2$ spaces:

$$\mathcal{C}^{\mathrm{EIR}} := \prod_{i=1}^{n} \left\{ \omega_i \in L^2(F_{-i}) : 0 \le \omega_i \le 1 \quad F_{-i}\text{-almost everywhere} \right\}.$$

We retain the notation $\mathcal{C}^{\mathrm{EIR}}$ for this quotient space; measurable representatives are used when constructing mechanisms. Each component is convex, norm closed, and bounded and hence weakly compact by the same results. Thus $\mathcal{C}^{\mathrm{EIR}}$ is weakly compact and convex, while its constant-rule subset is homeomorphic to the compact convex set $[0,1]^n$.

The critical-rent functional is well defined on these equivalence classes. Indeed, if $\delta_i := \inf_{[0,1]} f_i > 0$, then

$$\mathbb{E}_{-i}[R_i^{\boldsymbol{q}}(c_i(\boldsymbol{\theta}_{-i}), \boldsymbol{\theta}_{-i})] = \mathbb{E}\left[ \frac{\mathbb{1}\{\theta_i \le c_i(\boldsymbol{\theta}_{-i})\}}{f_i(\theta_i)} m_i(\boldsymbol{q}(\boldsymbol{\theta})) \right].$$

The multiplier is measurable and bounded by $1/\delta_i$. Thus, for each fixed $\boldsymbol{c}$, $\widetilde{W}_\kappa(\boldsymbol{q}, \boldsymbol{c})$ is affine and weakly continuous in $\boldsymbol{q}$. For each fixed $\boldsymbol{q} \in \widehat{\mathcal{Q}}$, equation (11) shows that the dependence

on $\boldsymbol{c}$, up to a constant, is

$$\sum_{i=1}^{n}(\kappa - w_i)\mathbb{E}_{-i}\left[R_i^{\boldsymbol{q}}(c_i(\boldsymbol{\theta}_{-i}), \boldsymbol{\theta}_{-i})\right].$$

Weak monotonicity of each own-type coefficient schedule makes $R_i^{\boldsymbol{q}}(\cdot, \boldsymbol{\theta}_{-i})$ convex. Boundedness of the coefficients makes this function uniformly Lipschitz. Consequently, replacing $c_i$ by $c_i'$ changes agent $i$'s critical-type term by at most

$$(\kappa - w_i)\max_k |m_i^k| \|c_i - c_i'\|_{L^1} \leq (\kappa - w_i)\max_k |m_i^k| \|c_i - c_i'\|_{L^2}.$$

The integral functional is therefore convex and norm continuous in $\boldsymbol{c}$, and hence weakly lower semicontinuous (Aliprantis and Border, 2006, Corollary 5.99).

The hypotheses of Sion's minimax theorem hold on $\widehat{\mathcal{Q}}$: the allocation rule ranges over a weakly compact convex set and enters affinely and continuously, while the critical-type argument ranges over a weakly compact convex set and enters convexly and lower semicontinuously. For either critical-type class, Sion (1958, Theorem 3.4) gives

$$\max_{\boldsymbol{q}\in\widehat{\mathcal{Q}}}\min_{\boldsymbol{c}}\widetilde{W}_\kappa(\boldsymbol{q}, \boldsymbol{c}) = \min_{\boldsymbol{c}}\max_{\boldsymbol{q}\in\widehat{\mathcal{Q}}}\widetilde{W}_\kappa(\boldsymbol{q}, \boldsymbol{c}).$$

The pointwise infimum over $\boldsymbol{c}$ is upper semicontinuous in $\boldsymbol{q}$, and the pointwise supremum over $\boldsymbol{q}$ is lower semicontinuous in $\boldsymbol{c}$. Compactness therefore gives primal and dual attainment in the relaxed problems.

*Recovery of pointwise incentive compatibility.*

For IR, let $\widehat{\boldsymbol{q}}$ be a relaxed maximizer and choose a Borel measurable, everywhere simplex-valued representative. Its interim coefficients admit bounded, weakly increasing versions $\widehat{M}_i : [0,1] \to [\min_k m_i^k, \max_k m_i^k]$. Let $N_i$ be the measurable $F_i$-null set of own types at which the representative's interim coefficient differs from $\widehat{M}_i$. For $x \in N_i$, choose a lottery $\boldsymbol{p}_i(x)$ with $m_i(\boldsymbol{p}_i(x)) = \widehat{M}_i(x)$, measurably in $x$: a mixture of alternatives with the smallest and largest own-type coefficients suffices. If these coefficients coincide, any fixed lottery suffices. Define $\boldsymbol{q}^B = \widehat{\boldsymbol{q}}$ when no report lies in its corresponding $N_i$; otherwise set $\boldsymbol{q}^B(\boldsymbol{\theta}) = \boldsymbol{p}_i(\theta_i)$ for the smallest index with $\theta_i \in N_i$. Independence implies that, for every own report $x$, opponents' exceptional reports have probability zero. Hence $\boldsymbol{q}^B$ has interim coefficient $\widehat{M}_i(x)$ at every $x$, while $\boldsymbol{q}^B = \widehat{\boldsymbol{q}}$ $F$-almost everywhere.

Normalize interim utility by

$$u_i(x) := \int_0^x \widehat{M}_i(s)\,ds - \min_{y\in[0,1]}\int_0^y \widehat{M}_i(s)\,ds,$$

and take $t_i^B(\boldsymbol{\theta}) = a_i(\boldsymbol{q}^B(\boldsymbol{\theta})) + \theta_i m_i(\boldsymbol{q}^B(\boldsymbol{\theta})) - u_i(\theta_i)$. Monotonicity of $\widehat{M}_i$ and the envelope formula give BIC and IR for all own types and reports. Its objective equals the relaxed IR maximum: the allocation rule agrees almost everywhere and its interim utility uses a worst-off normalization. The BIC–DIC construction in Lemma 1, using Gershkov et al. (2013, Theorem 1), supplies a pointwise DIC–IR mechanism with the same expected utilities and profit. Thus the original IR problem attains the relaxed value. This construction need not preserve $\widehat{\boldsymbol{q}}$ pointwise.

For EIR, the identity for a fixed allocation rule $\min_{\boldsymbol{\omega}} \widetilde{W}_\kappa(\boldsymbol{q}, \boldsymbol{\omega}) = \kappa\mathbb{E}[\Phi_\kappa(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{\theta})]$ holds on $\widehat{\mathcal{Q}}$ by the same almost-everywhere envelope calculation. Its relaxed value is at most $\kappa\mathbb{E}[\max_{\boldsymbol{p}\in\Delta(\mathcal{K})} \Phi_\kappa(\boldsymbol{p}, \boldsymbol{\theta})]$. Under Assumption 1, Lemma 3 provides a pointwise VSM rule in $\mathcal{Q}^{\mathrm{DIC}}$. Normalizing at each profilewise rent minimum gives a DIC–EIR mechanism attaining this upper bound. The lemma uses only the pointwise maximization problem, independently of the present proposition. Hence the original EIR problem also attains its relaxed value.

*Pairing primal and dual optimizers.*

Let $v$ be the common value for either participation class, $\boldsymbol{q}^* \in \mathcal{Q}^{\mathrm{DIC}}$ any original max–min optimizer, and $\boldsymbol{c}^*$ a relaxed dual minimizer. The equality of original and relaxed values gives

$$v = \min_{\boldsymbol{c}} \widetilde{W}_\kappa(\boldsymbol{q}^*, \boldsymbol{c}) \le \widetilde{W}_\kappa(\boldsymbol{q}^*, \boldsymbol{c}^*) \le \max_{\boldsymbol{q}\in\widehat{\mathcal{Q}}} \widetilde{W}_\kappa(\boldsymbol{q}, \boldsymbol{c}^*) = v.$$

Thus every such $\boldsymbol{q}^*$ pairs with $\boldsymbol{c}^*$ to form a saddle point, in particular over the original class of allocation rules $\mathcal{Q}^{\mathrm{DIC}}$. This proves the stated attainment and saddle-point claims without identifying almost-everywhere monotonicity with pointwise DIC. □

### A.2.5 Completion of Theorem 2

*Proof of Theorem 2(ii)–(iii).* Part (i) was established in Section A.2.3. Proposition 1 gives the fixed-rule values, and Proposition 2 establishes their attainment after optimization over DIC rules. Lemma 1 then gives

$$\begin{aligned} V_{\mathrm{BIC-IR}}(\kappa) = V_{\mathrm{DIC-IR}}(\kappa) &= \max_{\boldsymbol{q}\in\mathcal{Q}^{\mathrm{BIC}}} \kappa\mathbb{E}[\Phi_\kappa^{IR}(\boldsymbol{q}, \boldsymbol{\theta})] \\ &= \max_{\boldsymbol{q}\in\mathcal{Q}^{\mathrm{DIC}}} \kappa\mathbb{E}[\Phi_\kappa^{IR}(\boldsymbol{q}, \boldsymbol{\theta})]. \end{aligned} \tag{15}$$

Combining this with (9) proves all equalities in (5) and their attainment.

*Fixed-$\kappa$ comparison.* Lemma 1 gives

$$V_{\mathrm{BIC-IR}}(\kappa) = V_{\mathrm{DIC-IR}}(\kappa) = V_{\mathrm{BIC-EIR}}(\kappa).$$

Every DIC–EIR mechanism is DIC–IR feasible, which gives the weak inequality. For strictness, let $\boldsymbol{q}$ be a fixed-$\kappa$ DIC–EIR optimal allocation rule with $\Gamma_i(\boldsymbol{q}) > 0$ for some agent $i$, and suppose $\kappa > \max_j w_j$. Evaluate the DIC–IR objective in (5) at this rule. This allocation rule attains the DIC–EIR value, and $\kappa > \max_j w_j$ gives $\sum_j (\kappa - w_j)\Gamma_j(\boldsymbol{q}) \geq (\kappa - w_i)\Gamma_i(\boldsymbol{q}) > 0$, which raises the DIC–IR objective strictly above that value. □

### A.2.6 Supporting Weights under EAND

We next connect fixed-$\kappa$ optimality to the EAND-constrained problem. The supporting-weight result below assumes that the constrained optimum is attained; this does not follow from fixed-$\kappa$ attainment alone.

An $X$–$Y$ problem is *strictly EAND-feasible* if some $X$–$Y$ mechanism has strictly positive expected profit.

**Proposition 3** (EAND support)**.** *Maintain Assumption 1. If the DIC–EIR problem is strictly EAND-feasible and its EAND-constrained value is attained, then there exist $\lambda^* \geq 0$ and $\kappa^* := w_0 + \lambda^* \geq \max_{i \in \mathcal{N}} w_i$ such that every DIC–EIR–EAND optimal mechanism $(\boldsymbol{q}^*, \boldsymbol{t}^*)$ solves the fixed-$\kappa^*$ DIC–EIR problem and satisfies $\lambda^* \Pi(\boldsymbol{q}^*, \boldsymbol{t}^*) = 0$. Consequently, $\boldsymbol{q}^*(\boldsymbol{\theta})$ maximizes pointwise virtual surplus at $\kappa^*$ for $F$-almost every type profile $\boldsymbol{\theta}$.*

*Proof.* Randomization makes the attainable pairs $(\Pi, J)$ convex. The EAND-constrained value $J^*$ is finite because $\sum_i \mathbb{E}[U_i] + \Pi = S$, surplus is bounded, and EIR makes expected utilities nonnegative. The standard separating-hyperplane argument therefore supplies a supporting multiplier: separate the attainable pairs from the open set $\{(p, j) : p > 0,\ j > J^*\}$. The separating coefficients on profit and the objective are nonnegative, and strict EAND feasibility makes the latter strictly positive. Normalizing that coefficient to one yields $\lambda^* \geq 0$ such that

$$J(\boldsymbol{q}, \boldsymbol{t}) + \lambda^* \Pi(\boldsymbol{q}, \boldsymbol{t}) \leq J^*$$

for every DIC–EIR mechanism. At any EAND-constrained optimizer, $J = J^*$ and $\Pi \geq 0$, so complementary slackness holds and the inequality binds. Thus every such optimizer maximizes $W_{\kappa^*}$, where $\kappa^* := w_0 + \lambda^*$.

It remains to verify $\kappa^* \geq \max_i w_i$. If $\kappa^* < w_i$ for some agent $i$, lowering that agent's transfer by a report-independent constant $K > 0$ preserves DIC and EIR, raises the agent's utility by $K$, and lowers expected profit by $K$. The fixed-$\kappa^*$ objective then increases by $(w_i - \kappa^*)K$ and is unbounded, contradicting the existence of the fixed-$\kappa^*$ maximizer $(\boldsymbol{q}^*, \boldsymbol{t}^*)$. Therefore $\kappa^* \geq \max_{i \in \mathcal{N}} w_i$. Moreover, $\kappa^* > 0$: either some agent's weight is positive, or all agent weights vanish and the maintained nontriviality of the welfare weights gives

$w_0 > 0$. Theorem 2 now implies that $\boldsymbol{q}^*(\boldsymbol{\theta})$ maximizes pointwise virtual surplus at $\kappa^*$ almost everywhere. □

## A.3 Proofs for Section 4

We first translate positive efficient-rule rent gaps into strict signs of surplus-maximizing coefficients. The propagation proof transfers these signs to an IR optimizer through ironed virtual types. A supporting-weight argument then establishes strict Pareto improvement, and the final proof establishes positive efficient-rule rent gaps under conservation.

**Efficient allocation rules.** Extend the affine surplus function to $\boldsymbol{v} \in \mathbb{R}^n$ by

$$s(\boldsymbol{q}, \boldsymbol{v}) := \sum_i [a_i(\boldsymbol{q}) + v_i m_i(\boldsymbol{q})] - C(\boldsymbol{q}).$$

For each alternative $k$, let $\boldsymbol{e}^k$ be the degenerate lottery selecting $k$. Define the maximum-surplus function and its maximizing alternatives by

$$G(\boldsymbol{v}) := \max_{\boldsymbol{q} \in \Delta(\mathcal{K})} s(\boldsymbol{q}, \boldsymbol{v}), \qquad \mathcal{A}(\boldsymbol{v}) := \arg\max_{k \in \mathcal{K}} s(\boldsymbol{e}^k, \boldsymbol{v}).$$

These objects agree with the actual surplus problem on the type space. Their affine extension will also be evaluated at virtual types outside that space.

**Lemma 4** (Efficient rent gaps and maximizing coefficients)**.** *The allocation rule $\boldsymbol{q}^{\text{eff}}$ belongs to $\mathcal{Q}^{\text{DIC}}$, and its accumulated rent satisfies*

$$R_i^{\boldsymbol{q}^{\text{eff}}}(x, \boldsymbol{\theta}_{-i}) = G(x, \boldsymbol{\theta}_{-i}) - G(0, \boldsymbol{\theta}_{-i}). \tag{16}$$

*Consequently, its rent gaps are independent of efficient tie-breaking. Fix an agent $i$ with $\Gamma_i(\boldsymbol{q}^{\text{eff}}) > 0$. For every $z \in \mathbb{R}$, there exists $\boldsymbol{v}_{-i} \in (0,1)^{n-1}$ such that the coefficients $\{m_i^k : k \in \mathcal{A}(z, \boldsymbol{v}_{-i})\}$ are either all strictly positive or all strictly negative. This common strict sign persists in a neighborhood of $(z, \boldsymbol{v}_{-i})$. In addition, there exist $\boldsymbol{v}_{-i}^-, \boldsymbol{v}_{-i}^+ \in (0,1)^{n-1}$ and $\varepsilon_i^-, \varepsilon_i^+ \in (0,1)$ such that*

$$\begin{aligned} z \le \varepsilon_i^- \quad &\Longrightarrow \quad m_i^k < 0 \quad \text{for every } k \in \mathcal{A}(z, \boldsymbol{v}_{-i}^-), \\ z \ge 1 - \varepsilon_i^+ \quad &\Longrightarrow \quad m_i^k > 0 \quad \text{for every } k \in \mathcal{A}(z, \boldsymbol{v}_{-i}^+). \end{aligned} \tag{17}$$

*Proof.* Fix $\boldsymbol{v}_{-i}$, take $x < y$, and choose any $k_x \in \mathcal{A}(x, \boldsymbol{v}_{-i})$ and $k_y \in \mathcal{A}(y, \boldsymbol{v}_{-i})$. Adding their

maximizing inequalities gives

$$(y - x)(m_i^{k_y} - m_i^{k_x}) \geq 0. \tag{18}$$

Thus every pointwise efficient selection has a nondecreasing own-type coefficient. A deterministic selection with fixed tie-breaking is measurable because the alternative set is finite and each surplus function is continuous, so $\boldsymbol{q}^{\text{eff}} \in \mathcal{Q}^{\text{DIC}}$. For fixed opponents' types, $G(x, \boldsymbol{\theta}_{-i})$ is a finite maximum of affine functions of $x$. It is convex and absolutely continuous, and at every differentiability point its derivative equals the own-type coefficient of every maximizing allocation. Integrating this derivative proves (16). It also proves that accumulated rents and rent gaps are independent of efficient tie-breaking.

By (16), the profilewise worst-off types under efficiency are $\arg\min_{x\in[0,1]} G(x, \boldsymbol{\theta}_{-i})$. Theorem 1 and $\Gamma_i(\boldsymbol{q}^{\text{eff}}) > 0$ imply that, for every $x \in [0, 1]$, there is a positive-probability set of opponents' profiles at which $x$ is not a minimizer. Because types are atomless, a profile in that set can be chosen in $(0, 1)^{n-1}$.

For $z \in (0, 1)$, choose such a profile $\boldsymbol{v}_{-i}$. The subdifferential of the convex function $G(\cdot, \boldsymbol{v}_{-i})$ at $z$ is the convex hull of $\{m_i^k : k \in \mathcal{A}(z, \boldsymbol{v}_{-i})\}$. Since $z$ is not a minimizer, this hull excludes zero, so all maximizing coefficients have the same strict sign. At $z = 0$, choose a profile $\boldsymbol{v}_{-i}^-$ where zero is not a minimizer. Every maximizing coefficient must be negative: a maximizing alternative with a nonnegative coefficient would imply $G(x, \boldsymbol{v}_{-i}^-) \geq G(0, \boldsymbol{v}_{-i}^-)$ for every $x \geq 0$. At $z = 1$, the analogous argument gives a profile $\boldsymbol{v}_{-i}^+$ where every maximizing coefficient is positive.

At any of these profiles, all alternatives with a zero coefficient or the opposite sign lie strictly below the maximum. Finiteness and continuity preserve this strict exclusion in an open neighborhood. In particular, the negative sign at $(0, \boldsymbol{v}_{-i}^-)$ persists at $(\varepsilon_i^-, \boldsymbol{v}_{-i}^-)$ for some $\varepsilon_i^- > 0$, and the positive sign at $(1, \boldsymbol{v}_{-i}^+)$ persists at $(1 - \varepsilon_i^+, \boldsymbol{v}_{-i}^+)$ for some $\varepsilon_i^+ > 0$. Equation (18) extends these signs to every smaller or larger own coordinate, respectively, proving (17). This also establishes the stated sign property for $z \notin [0, 1]$. Every maximizing lottery is supported on maximizing alternatives, so it inherits their strict sign. □

**Lemma 5** (Propagation of an efficient rent gap)**.** *Maintain Assumption 1. Fix $\kappa > 0$ and $0 \leq w_j \leq \kappa$ for every agent $j$. Let $\boldsymbol{q}^*$ be the allocation rule of any DIC–IR mechanism maximizing $W_\kappa$. Then, for every agent $i$,*

$$\Gamma_i(\boldsymbol{q}^{\text{eff}}) > 0 \quad \Longrightarrow \quad \Gamma_i(\boldsymbol{q}^*) > 0.$$

The proof represents an IR-optimal allocation as a surplus maximizer at ironed virtual

types and uses the positive efficient-rule gap to rule out a common ex post worst-off type. It covers boundary types and nonunique maximizers.

*Proof of Lemma 5.* Write $\alpha_j = (\kappa - w_j)/\kappa$. Fix an agent $i$ with $\Gamma_i(\boldsymbol{q}^{\text{eff}}) > 0$.

*Critical types and ironing.* Proposition 2 pairs the given allocation rule $\boldsymbol{q}^*$ with a vector of constant critical types $\boldsymbol{c}$ at an IR saddle point. Consequently, $\boldsymbol{q}^*$ maximizes

$$L_{\boldsymbol{c}}(\boldsymbol{q}) := \mathbb{E}\left[\sum_j \{a_j(\boldsymbol{q}(\boldsymbol{\theta})) + \psi_j(\theta_j; c_j) m_j(\boldsymbol{q}(\boldsymbol{\theta}))\} - C(\boldsymbol{q}(\boldsymbol{\theta}))\right] \tag{19}$$

over $\mathcal{Q}^{\text{DIC}}$.

We use the critical-type ironing construction of Loertscher and Wasser (2019, Section 3.1). The notation $\bar{\psi}_j$ suppresses dependence on the critical type $c_j$ and the fixed welfare weights. If $\alpha_j = 0$, set $\bar{\psi}_j(x) = x$. If $\alpha_j > 0$ and $c_j = 0$, set $\bar{\psi}_j = \psi_j^B$; if $c_j = 1$, set $\bar{\psi}_j = \psi_j^S$. These schedules agree with the corresponding raw schedules almost everywhere. For $\alpha_j > 0$ and $0 < c_j < 1$, suppress the index and define

$$\bar{\psi}_z(x) = \max\{\psi^B(x), \min\{z, \psi^S(x)\}\}.$$

Choose $z$ to satisfy

$$\int_0^1 \bar{\psi}_z(x) f(x)\, dx = \int_0^1 \psi(x; c) f(x)\, dx. \tag{20}$$

Such a $z$ exists uniquely in $(\psi^B(c), \psi^S(c))$. Indeed, the left-hand side is continuous and strictly increasing on this interval. At $z = \psi^B(c)$ it is strictly smaller than the right-hand side, and at $z = \psi^S(c)$ it is strictly larger. Strictness follows from positive density and continuity of the strictly increasing schedules $\psi^B$ and $\psi^S$.

Write $\bar{\psi} = \bar{\psi}_z$. The schedule is continuous and nondecreasing, and is constant on the nonempty interval

$$I = \{x \in (0,1) : \psi^B(x) < z < \psi^S(x)\}, \qquad c \in I.$$

Define $D(x) = \int_0^x [\psi(s; c) - \bar{\psi}(s)] f(s)\, ds$. The raw schedule exceeds the ironed schedule on $I \cap (0, c)$ and lies below it on $I \cap (c, 1)$, while the two agree almost everywhere outside $I$. Equation (20) therefore implies $D(0) = D(1) = 0$, $D > 0$ on $I$, and $D = 0$ outside $I$. For every bounded nondecreasing coefficient schedule $m$, Stieltjes integration by parts gives

$$\int_0^1 [\bar{\psi}(x) - \psi(x; c)] m(x) f(x)\, dx = \int_{(0,1)} D(x)\, dm(x) \geq 0. \tag{21}$$

In particular, equality holds when $m$ is constant on $I$. A right-continuous version of $m$ gives the same integrals.

*Attainment of the ironing bound.* Applying this inequality conditional on opponents' reports, and using independence, gives, for every $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{DIC}}$,

$$L_{\boldsymbol{c}}(\boldsymbol{q}) \leq \mathbb{E}[s(\boldsymbol{q}(\boldsymbol{\theta}), \bar{\boldsymbol{\psi}}(\boldsymbol{\theta}))] \leq \mathbb{E}[G(\bar{\boldsymbol{\psi}}(\boldsymbol{\theta}))].$$

Both bounds are attained by a single DIC allocation rule. To see this, select a deterministic alternative maximizing $s(\cdot, \bar{\boldsymbol{\psi}}(\boldsymbol{\theta}))$, using a fixed ordering of alternatives to break ties. The selection is measurable. Monotonicity of the ironed schedules and (18) imply DIC; when an agent's ironed type is constant, the selected allocation is unchanged, so every ironing inequality binds. It follows that every maximizer of $L_{\boldsymbol{c}}$, including the given $\boldsymbol{q}^*$, attains both bounds. Thus

$$\boldsymbol{q}^*(\boldsymbol{\theta}) \in \arg\max_{\boldsymbol{q} \in \Delta(\mathcal{K})} s(\boldsymbol{q}, \bar{\boldsymbol{\psi}}(\boldsymbol{\theta})) \qquad \text{for } F\text{-almost every } \boldsymbol{\theta}. \tag{22}$$

No uniqueness of the optimal allocation rule is required.

*The range of ironed types.* For every agent $j$, the schedule $\bar{\psi}_j$ is continuous on $[0,1]$ and satisfies $\psi_j^B \leq \bar{\psi}_j \leq \psi_j^S$. Since $\psi_j^S(0) = 0$ and $\psi_j^B(1) = 1$,

$$\bar{\psi}_j(0) \leq 0, \qquad \bar{\psi}_j(1) \geq 1.$$

Consequently, its range contains $(0,1)$. Continuity and full support imply that, for every $v_j \in (0,1)$ and $\eta > 0$,

$$\Pr\{|\bar{\psi}_j(\theta_j) - v_j| < \eta\} > 0. \tag{23}$$

*Excluding a common worst-off type.* Suppose, toward a contradiction, that $\Gamma_i(\boldsymbol{q}^*) = 0$. Theorem 1 supplies a common ex post worst-off type $x^\circ \in [0,1]$. This type need not coincide with the saddle-point critical type $c_i$.

DIC makes $x \mapsto m_i(\boldsymbol{q}^*(x, \boldsymbol{\theta}_{-i}))$ nondecreasing at every opponents' profile. For almost every such profile, accumulated rent is minimized at $x^\circ$, so

$$\begin{aligned} m_i(\boldsymbol{q}^*(x, \boldsymbol{\theta}_{-i})) &\leq 0 \quad \text{for almost every } x < x^\circ, \\ m_i(\boldsymbol{q}^*(x, \boldsymbol{\theta}_{-i})) &\geq 0 \quad \text{for almost every } x > x^\circ. \end{aligned} \tag{24}$$

For example, a positive coefficient at an interior point below $x^\circ$, together with monotonicity, would make accumulated rent strictly increase from that point to $x^\circ$, contradicting minimization. The other inequality follows symmetrically.

We use the following neighborhood argument in each case. If all surplus-maximizing

allocations have a common strict coefficient sign near $(z, \boldsymbol{v}_{-i})$, continuity of $\bar{\psi}_i$ transfers that sign to any sufficiently close nonempty open interval of own types whose ironed types lie in the neighborhood. Equation (23), full support, and independence give positive probability to the event that own type lies in this interval and opponents' ironed types lie near $\boldsymbol{v}_{-i}$. On this event, (22) imposes the strict sign almost surely.

First suppose $0 < x^\circ < 1$, and set $z = \bar{\psi}_i(x^\circ)$. By Lemma 4, there exists $\boldsymbol{v}_{-i} \in (0,1)^{n-1}$ and a neighborhood of $(z, \boldsymbol{v}_{-i})$ in which all surplus-maximizing allocations have own-type coefficients of the same strict sign for agent $i$. If that sign is positive, continuity of $\bar{\psi}_i$ supplies a nonempty open interval $J \subset (0, x^\circ)$, sufficiently close to $x^\circ$, such that $\bar{\psi}_i(J)$ remains in the relevant own-coordinate neighborhood. The neighborhood argument therefore gives a strictly positive coefficient on a positive-probability event below $x^\circ$, contradicting (24). If the sign is negative, use instead an interval $J \subset (x^\circ, 1)$ to obtain the opposite contradiction.

Next suppose $x^\circ = 0$. Set $z_0 = \bar{\psi}_i(0) \leq 0$. The boundary-sign conclusion of Lemma 4 supplies $\boldsymbol{v}_{-i}^- \in (0,1)^{n-1}$ such that all surplus-maximizing allocations at $(z_0, \boldsymbol{v}_{-i}^-)$ have strictly negative coefficients for agent $i$. This strict sign persists in a neighborhood. Continuity of $\bar{\psi}_i$ therefore supplies a nonempty interval of positive own types close to zero whose ironed types lie in that neighborhood. The neighborhood argument again gives a negative coefficient on a positive-probability event. This contradicts the nonnegative coefficient required by (24) when $x^\circ = 0$.

Finally, suppose $x^\circ = 1$. Now $z_1 = \bar{\psi}_i(1) \geq 1$, and Lemma 4 supplies an opponents' profile at which every surplus-maximizing allocation has a strictly positive coefficient. The same neighborhood argument, using own types just below one, contradicts the nonpositive coefficient required by (24).

All cases contradict the existence of a common ex post worst-off type. Hence $\Gamma_i(\boldsymbol{q}^*) > 0$. The argument applies to every agent with a positive efficient-rule rent gap, including agents with $\alpha_i = 0$, or equivalently $w_i = \kappa$. □

*Proof of Theorem 3. Supporting weights.* Suppose the benchmark admits no strict Pareto improvement. Let $P \subset \mathbb{R}^{n+1}$ be the set of vectors of designer profit and agents' expected utilities attainable by DIC–IR mechanisms, temporarily omitting EAND, and let $v^\circ \in P$ be the benchmark vector. Lotteries preserve DIC and IR, so $P$ is convex. It is disjoint from the open convex set $O := v^\circ + \mathbb{R}^{n+1}_{++}$: a vector in their intersection would have profit strictly above the benchmark's nonnegative profit and would therefore also satisfy EAND.

Separation gives a nonzero vector $\boldsymbol{\lambda}$ and a number $a$ such that

$$\boldsymbol{\lambda} \cdot v \leq a \leq \boldsymbol{\lambda} \cdot o \qquad (v \in P,\ o \in O).$$

No closedness of $P$ is needed because $O$ is open. Its unboundedness in every positive coordinate direction implies $\boldsymbol{\lambda} \geq 0$. Since $v^\circ \in P$ and $v^\circ + \varepsilon \mathbf{1} \in O$ for every $\varepsilon > 0$, letting $\varepsilon \downarrow 0$ gives $a = \boldsymbol{\lambda} \cdot v^\circ$. The benchmark consequently maximizes

$$\lambda_0 \Pi(\boldsymbol{q}, \boldsymbol{t}) + \sum_i \lambda_i \mathbb{E}[U_i]$$

over DIC–IR mechanisms. A report-independent subsidy of $K > 0$ to agent $i$ preserves DIC and IR and changes this objective by $(\lambda_i - \lambda_0)K$. Hence $0 \leq \lambda_i \leq \lambda_0$ for every $i$, and nontriviality of the weights gives $\lambda_0 > 0$.

*Weights outside $Z$.* Set $\kappa = \lambda_0$ and $w_j = \lambda_j$. For every $i \notin Z$, either the benchmark rent gap is positive or the efficient-rule rent gap is positive. In the latter case, Lemma 5 applies to the benchmark, which is a DIC–IR optimizer for these weights, and gives $\Gamma_i(\boldsymbol{q}^\circ) > 0$. If $\lambda_i < \lambda_0$, weighted optimality requires agent $i$'s expected utility to equal its minimum under DIC–IR at the benchmark rule. Indeed, otherwise a constant increase in his payment could lower his utility to that minimum while preserving DIC and IR and increasing the objective. The benchmark is EIR, however, so Theorem 1 places its expected utility at least $\Gamma_i(\boldsymbol{q}^\circ) > 0$ above that minimum. Hence $\lambda_i = \lambda_0$ for every $i \notin Z$. The supporting objective becomes

$$\lambda_0 S(\boldsymbol{q}) - \sum_{i \in Z} (\lambda_0 - \lambda_i) \mathbb{E}[U_i].$$

*Preserving interim utilities.* For $i \in Z$, write $u_i^\circ(\theta_i) := \mathbb{E}_{-i}[U_i^\circ(\theta_i, \boldsymbol{\theta}_{-i})]$. Under $\widehat{\boldsymbol{q}}$, set

$$\widehat{U}_i(\boldsymbol{\theta}) := u_i^\circ(0) + \int_0^{\theta_i} m_i(\widehat{\boldsymbol{q}}(s, \boldsymbol{\theta}_{-i}))\, ds, \qquad \widehat{t}_i(\boldsymbol{\theta}) := a_i(\widehat{\boldsymbol{q}}(\boldsymbol{\theta})) + \theta_i m_i(\widehat{\boldsymbol{q}}(\boldsymbol{\theta})) - \widehat{U}_i(\boldsymbol{\theta}).$$

Monotonicity of the own-type coefficient makes these transfers DIC. By (7) and the benchmark envelope formula,

$$\mathbb{E}_{-i}[\widehat{U}_i(\theta_i, \boldsymbol{\theta}_{-i})] = u_i^\circ(0) + \int_0^{\theta_i} M_i^{\boldsymbol{q}^\circ}(s)\, ds = u_i^\circ(\theta_i) \geq 0.$$

Thus the transfers preserve these agents' interim utility functions and satisfy IR. For agents outside $Z$, use the IR worst-off normalization under $\widehat{\boldsymbol{q}}$. All resulting transfers are measurable and integrable because coefficients are bounded and the benchmark interim utilities are finite.

*Contradiction to weighted optimality.* This DIC–IR mechanism preserves expected utilities for agents in $Z$, so its supporting objective exceeds the benchmark's by

$$\lambda_0 \big[ S(\widehat{\boldsymbol{q}}) - S(\boldsymbol{q}^\circ) \big] > 0,$$

contradicting optimality. This comparison does not require EAND for the constructed mechanism: separation was over all DIC–IR mechanisms. The contradiction establishes a strict Pareto improvement, which satisfies EAND because its profit exceeds the benchmark's nonnegative profit. □

*Proof of Corollary 1.* Fix an agent $i$. Let $G$ be the maximum-surplus function defined above. For any $t \in (0,1)$, conservation gives

$$G(x, t\mathbf{1}_{-i}) = \max_{k \in \mathcal{K}} (x - t) m_i^k.$$

Because $\min_k m_i^k < 0 < \max_k m_i^k$, this expression equals zero at $x = t$ and is strictly positive at every $x \neq t$.

Fix any $x \in [0,1]$, and choose $t \in (0,1)$ with $t \neq x$. Then

$$G(x, t\mathbf{1}_{-i}) > G(t, t\mathbf{1}_{-i}).$$

Continuity preserves this strict inequality on a neighborhood of $t\mathbf{1}_{-i}$, which has positive probability by full support. By (16), $x$ therefore fails to minimize accumulated rent on a positive-probability set of opponents' profiles. Since this holds for every $x$, there is no common ex post worst-off type under efficiency. Theorem 1 implies $\Gamma_i(\boldsymbol{q}^{\text{eff}}) > 0$. Thus $Z = \varnothing$. For any inefficient benchmark, choose $\widehat{\boldsymbol{q}} = \boldsymbol{q}^{\text{eff}}$: expected surplus strictly increases and (7) is vacuous. Theorem 3 proves the conclusion. □

## A.4 Proofs for Section 5

The first lemma bounds agent 1's payoff in both regimes; the second verifies that the common mechanism constructed in Section 5.2 attains both bounds and balances the budget conditional on every public type.

**Lemma 6** (Bounds from a common allocation rule)**.** *The allocation rule $\boldsymbol{q}^1$ is measurable, belongs to $\mathcal{Q}^{\text{DIC}}$, and is independent of $F_1$. For every $x \in [0,1]$ and every $(\boldsymbol{q}_x, \boldsymbol{t}_x) \in \mathcal{M}^T(x)$,*

$$\mathbb{E}_{-1}[U_{1,x}(\boldsymbol{\theta}_{-1})] \leq \mathbb{E}_{-1}\left[\Phi^1(\boldsymbol{q}_x(\boldsymbol{\theta}_{-1}), x, \boldsymbol{\theta}_{-1})\right] \leq \mathbb{E}_{-1}[V^1(x; \boldsymbol{\theta}_{-1})].$$

*For every $(\boldsymbol{q}, \boldsymbol{t}) \in \mathcal{M}^{EA}$, $\mathbb{E}[U_1(\boldsymbol{\theta})] \leq \mathbb{E}[V^1(\theta_1; \boldsymbol{\theta}_{-1})]$.*

*Proof. Common allocation.* Because $\alpha_1 = 0$, agent 1's virtual value and virtual cost both equal his type. Hence $\Phi^1$ and its maximizing correspondence do not depend on $F_1$. Section 5.2 chooses the measurable selection $\boldsymbol{q}^1$ independently of $F_1$. Lemma 3 ensures that its own-type coefficient schedules are weakly increasing, so $\boldsymbol{q}^1 \in \mathcal{Q}^{\text{DIC}}$.

*Transparent bound.* Fix $x$ and a transparent feasible contract $(\boldsymbol{q}_x, \boldsymbol{t}_x)$. For each reporting agent $i \neq 1$, let $U_{i,x}(y, \boldsymbol{\theta}_{-\{1,i\}})$ denote type $y$'s truthful payoff when the remaining reports are $\boldsymbol{\theta}_{-\{1,i\}}$. Throughout this proof and the conditional-balance proof below, $\mathbb{E}_i$ integrates only $\theta_i$ with respect to $F_i$, holding $x$ and $\boldsymbol{\theta}_{-\{1,i\}}$ fixed. The DIC envelope formula and integration by parts give

$$\mathbb{E}_i[t_{i,x}(\boldsymbol{\theta}_{-1})] = \mathbb{E}_i[\rho_i(\boldsymbol{q}_x(\boldsymbol{\theta}_{-1}), \theta_i)] - \min_{y\in[0,1]} U_{i,x}(y, \boldsymbol{\theta}_{-\{1,i\}}). \tag{25}$$

EIR makes the final term nonnegative. Taking expectations over the remaining agents therefore yields $\mathbb{E}_{-1}[t_{i,x}] \leq \mathbb{E}_{-1}[\rho_i(\boldsymbol{q}_x(\boldsymbol{\theta}_{-1}), \theta_i)]$. Conditional no deficit and these transfer bounds imply

$$\begin{aligned}
\mathbb{E}_{-1}[U_{1,x}] &= \mathbb{E}_{-1}\left[a_1(\boldsymbol{q}_x(\boldsymbol{\theta}_{-1})) + x m_1(\boldsymbol{q}_x(\boldsymbol{\theta}_{-1})) - t_{1,x}\right] \\
&\leq \mathbb{E}_{-1}\left[a_1(\boldsymbol{q}_x(\boldsymbol{\theta}_{-1})) + x m_1(\boldsymbol{q}_x(\boldsymbol{\theta}_{-1})) + \sum_{i\neq 1} \rho_i(\boldsymbol{q}_x(\boldsymbol{\theta}_{-1}), \theta_i) - C(\boldsymbol{q}_x(\boldsymbol{\theta}_{-1}))\right] \\
&= \mathbb{E}_{-1}\left[\Phi^1(\boldsymbol{q}_x(\boldsymbol{\theta}_{-1}), x, \boldsymbol{\theta}_{-1})\right] \\
&\leq \mathbb{E}_{-1}[V^1(x; \boldsymbol{\theta}_{-1})],
\end{aligned}$$

which proves the transparent bound.

*Ex ante bound.* Fix an ex ante feasible mechanism $(\boldsymbol{q}, \boldsymbol{t})$. EAND gives $\mathbb{E}[U_1(\boldsymbol{\theta})] \leq W_{\kappa=1}(\boldsymbol{q}, \boldsymbol{t})$. Theorem 2 and the definition of $\boldsymbol{q}^1$ then give

$$W_{\kappa=1}(\boldsymbol{q}, \boldsymbol{t}) \leq \mathbb{E}\left[\Phi^1(\boldsymbol{q}^1(\boldsymbol{\theta}), \boldsymbol{\theta})\right] = \mathbb{E}[V^1(\theta_1; \boldsymbol{\theta}_{-1})].$$

This proves the ex ante bound. □

**Lemma 7** (Conditional balance)**.** *There exists an integrable transfer rule $\boldsymbol{t}^*$ such that $(\boldsymbol{q}^1, \boldsymbol{t}^*)$ is DIC–EIR, $U_1^*(x, \boldsymbol{\theta}_{-1}) = V^1(x; \boldsymbol{\theta}_{-1})$ for every $x \in [0,1]$ and $\boldsymbol{\theta}_{-1}$, and*

$$\mathbb{E}_{-1}\left[\sum_{i=1}^{n} t_i^*(x, \boldsymbol{\theta}_{-1}) - C(\boldsymbol{q}^1(x, \boldsymbol{\theta}_{-1}))\right] = 0.$$

*Consequently, $(\boldsymbol{q}^1, \boldsymbol{t}^*) \in \mathcal{M}^{EA}$, and its $x$-slice belongs to $\mathcal{M}^T(x)$ for every $x$.*

*Proof. Reporting agents' constraints.* Use the transfer rule $\boldsymbol{t}^*$ constructed in Section 5.2. For every $i \neq 1$, it gives agent $i$ the VSM transfer under the EIR worst-off normalization for $\boldsymbol{q}^1$. Theorem 2 therefore ensures these agents' DIC and EIR constraints. The resulting transfers, including agent 1's allocation cost net of the other agents' virtual contributions, are measurable and integrable.

*Agent 1's incentives and participation.* Substitution into agent 1's payoff gives

$$\begin{aligned} U_1^*(x, \boldsymbol{\theta}_{-1}) &= a_1(\boldsymbol{q}^1(x, \boldsymbol{\theta}_{-1})) + x m_1(\boldsymbol{q}^1(x, \boldsymbol{\theta}_{-1})) - t_1^*(x, \boldsymbol{\theta}_{-1}) \\ &= \Phi^1(\boldsymbol{q}^1(x, \boldsymbol{\theta}_{-1}), x, \boldsymbol{\theta}_{-1}) \\ &= V^1(x; \boldsymbol{\theta}_{-1}). \end{aligned}$$

Because the status quo satisfies $\Phi^1(\boldsymbol{q}^0, x, \boldsymbol{\theta}_{-1}) = 0$, pointwise maximization gives $V^1(x; \boldsymbol{\theta}_{-1}) \geq 0$, establishing agent 1's EIR constraint.

If agent 1's true type is $x$ and he reports $\hat{x}$, the definition of $t_1^*$ gives him

$$a_1(\boldsymbol{q}^1(\hat{x}, \boldsymbol{\theta}_{-1})) + x m_1(\boldsymbol{q}^1(\hat{x}, \boldsymbol{\theta}_{-1})) - t_1^*(\hat{x}, \boldsymbol{\theta}_{-1}) = \Phi^1(\boldsymbol{q}^1(\hat{x}, \boldsymbol{\theta}_{-1}), x, \boldsymbol{\theta}_{-1}) \leq V^1(x; \boldsymbol{\theta}_{-1}),$$

which is his truthful payoff. Thus agent 1 also satisfies DIC, and $(\boldsymbol{q}^1, \boldsymbol{t}^*)$ is DIC–EIR.

*Conditional balance.* Worst-off normalization sets the final term in (25) to zero. Hence, for every $i \neq 1$ and each fixed $x$ and $\boldsymbol{\theta}_{-\{1,i\}}$,

$$\mathbb{E}_i \left[ t_i^*(x, \boldsymbol{\theta}_{-1}) \right] = \mathbb{E}_i \left[ \rho_i(\boldsymbol{q}^1(x, \boldsymbol{\theta}_{-1}), \theta_i) \right].$$

The definition of $t_1^*$ gives the pointwise accounting identity

$$\sum_{i=1}^{n} t_i^*(x, \boldsymbol{\theta}_{-1}) - C(\boldsymbol{q}^1(x, \boldsymbol{\theta}_{-1})) = \sum_{i \neq 1} \left[ t_i^*(x, \boldsymbol{\theta}_{-1}) - \rho_i(\boldsymbol{q}^1(x, \boldsymbol{\theta}_{-1}), \theta_i) \right].$$

Independence and iterated expectations therefore give the conditional-balance equality in the lemma for every $x$. Hence the mechanism satisfies EAND. For each $x$, the reporting agents $i \neq 1$ retain DIC and EIR, and conditional balance makes the $x$-slice feasible in $\mathcal{M}^T(x)$. □

*Proof of Theorem 4.* Let $(\boldsymbol{q}^1, \boldsymbol{t}^*)$ be the mechanism supplied by Lemma 7. Fix $x$. Its $x$-slice is feasible in $\mathcal{M}^T(x)$ and gives agent 1 expected payoff $\mathbb{E}_{-1}[V^1(x; \boldsymbol{\theta}_{-1})]$. The transparent bound in Lemma 6 therefore gives $v^T(x) = \mathbb{E}_{-1}[V^1(x; \boldsymbol{\theta}_{-1})]$ for every $x$.

The mechanism $(\boldsymbol{q}^1, \boldsymbol{t}^*)$ is also feasible in $\mathcal{M}^{EA}$ and gives agent 1 expected payoff $\mathbb{E}[V^1(\theta_1; \boldsymbol{\theta}_{-1})]$. The ex ante bound in Lemma 6 shows that it attains $V^{EA}$. Consequently,

$$\begin{aligned} V^{EA} &= \mathbb{E}[V^1(\theta_1; \boldsymbol{\theta}_{-1})] \\ &= \int_0^1 \mathbb{E}_{-1}[V^1(x; \boldsymbol{\theta}_{-1})] \, dF_1(x) \\ &= \int_0^1 v^T(x) \, dF_1(x) = V^T. \end{aligned}$$

Setting $\boldsymbol{q}^* := \boldsymbol{q}^1$ completes the proof. $\square$

# References


Charalambos D Aliprantis and Kim C Border. *Infinite dimensional analysis: a hitchhiker's guide*. Springer, 2006.

Pasha Andreyanov, Junrok Park, and Tomasz Sadzik. Optimal robust double auction. Working paper, 2023.

Kim C Border. Reduced form auctions revisited. *Economic Theory*, 31(1):167–181, 2007.

T. Börgers. *An Introduction to the Theory of Mechanism Design*. Oxford University Press, New York, 2015.

Isabelle Brocas. Optimal allocation mechanisms with type-dependent negative externalities. *Theory and decision*, 75(3):359–387, 2013.

Yi-Chun Chen, Wei He, Jiangtao Li, and Yeneng Sun. Equivalence of stochastic and deterministic mechanisms. *Econometrica*, 87(4):1367–1390, 2019. doi: 10.3982/ECTA14698.

P Cramton, Robert Gibbons, and Paul Klemperer. Dissolving a partnership efficiently. *Econometrica*, 55(3):615–632, 1987.

Karsten Fieseler, Thomas Kittsteiner, and Benny Moldovanu. Partnerships, lemons, and efficient trade. *Journal of Economic Theory*, 113(2):223–234, 2003.

Stefano Galavotti, Nozomu Muto, and Daisuke Oyama. On efficient partnership dissolution under ex post individual rationality. *Economic Theory*, 48(1):87–123, 2011.

Alex Gershkov, Jacob K Goeree, Alexey Kushnir, Benny Moldovanu, and Xianwen Shi. On the equivalence of bayesian and dominant strategy implementation. *Econometrica*, 81(1): 197–220, 2013.

Bruno Jullien. Participation constraints in adverse selection models. *Journal of Economic Theory*, 93:1–47, 2000.

Georgia Kosmopoulou. Payoff equivalence between bayesian and ex post individually rational dominant strategy mechanisms. *Economic Theory*, 13(1):229–237, 1999. doi: 10.1007/s001990050251.

Tracy R Lewis and David E.M Sappington. Countervailing incentives in agency problems. *Journal of Economic Theory*, 49(1):294–313, 1989.

Simon Loertscher and Leslie M. Marx. Complete and incomplete information bargaining in a model with independent private values. Working paper, January 30, 2026, 2026. URL https://simonloertscher.net/wp-content/uploads/2026/09/Loertscher.Marx_.BAP_.2026.pdf.

Simon Loertscher and Ellen V Muir. Optimal hotelling auctions. Working Paper, 2026.

Simon Loertscher and Cédric Wasser. Optimal structure and dissolution of partnerships. *Theoretical Economics*, 14:1063–1114, 2019.

Hu Lu and Jacques Robert. Optimal trading mechanisms with ex ante unidentified traders. *Journal of Economic Theory*, 97(1):50–80, 2001.

Giovanni Maggi and Andres Rodriguez-Clare. On the countervailing incentives. *Journal of Economic Theory*, 66(1):238–263, 1995.

Louis Makowski and Claudio Mezzetti. Bayesian and weakly robust first best mechanisms: Characterizations. *Journal of Economic Theory*, 64(2):500–519, 1994. doi: 10.1006/jeth.1994.1079.

A. M. Manelli and D. R. Vincent. Bayesian and dominant-strategy implementation in the independent private values model. *Econometrica*, 78:1905–1938, 2011.

Roger Myerson and Mark Satterthwaite. Efficient mechanisms for bilateral trading. *Journal of Economic Theory*, 29(2):265–281, 1983.

Tymofiy Mylovanov and Thomas Tröger. Mechanism design by an informed principal: Private values with transferable utility. *Review of Economic Studies*, 81(4):1668–1707, 2014.

Maurice Sion. On general minimax theorems. *Pacific Journal of Mathematics*, 8(1):171–176, 1958.

William Vickrey. Counterspeculation, auction, and competitive sealed tenders. *Journal of Finance*, 16(1):8–37, 1961.

Steven R. Williams. Efficient performance in two agent bargaining. *Journal of Economic Theory*, 41(1):154–172, 1987.

# Supplemental Appendix

*Dominant strategies, ex post individual rationality, and optimal mechanisms*

## S1 Examples and Partnership Calculations

### S1.1 The Hotelling Example

We verify Example 1, using its types, values, and zero allocation cost. Assignment probabilities satisfy $q_{i\ell} \geq 0$ and $\sum_{i,\ell} q_{i\ell} \leq 1$, with any residual probability assigned to no sale. Payments and utilities below are expected over the mechanism's lotteries, as in the general model. The independent uniform types satisfy full support and two-sided regularity, and both buyers face countervailing incentives.

**Proposition 4** (Revenue optimality without strict Pareto improvement)**.** *In this Hotelling environment, there exists a DIC–EIR–EAND mechanism that exhausts supply at every profile and is inefficient on a set of positive probability. Its expected revenue is* $17/24$*, the maximum over all DIC–IR mechanisms. For each buyer, both its rent gap and the efficient allocation rule's rent gap are zero. Consequently, no DIC–IR–EAND mechanism strictly improves every participant's expected payoff and the seller's expected profit.*

*Proof. An upper bound for every DIC–IR mechanism.* Consider any such mechanism, with interim utility $u_i(x) := \mathbb{E}_{-i}[U_i(x, \boldsymbol{\theta}_{-i})]$ and interim coefficient $M_i^{\boldsymbol{q}}(x) := \mathbb{E}_{-i}[m_i(\boldsymbol{q}(x, \boldsymbol{\theta}_{-i}))]$. The envelope condition gives

$$u_i(x) = u_i(1/2) + \int_{1/2}^{x} M_i^{\boldsymbol{q}}(s)\, ds.$$

Integrating under the uniform distribution and using $u_i(1/2) \geq 0$ yields

$$\mathbb{E}[U_i] \geq -\int_0^{1/2} x M_i^{\boldsymbol{q}}(x)\, dx + \int_{1/2}^{1} (1-x) M_i^{\boldsymbol{q}}(x)\, dx.$$

Subtracting these utility bounds from expected surplus gives

$$\Pi(\boldsymbol{q}, \boldsymbol{t}) \leq \mathbb{E}\left[\sum_i \{q_{i0}(\boldsymbol{\theta}) + \psi(\theta_i) m_i(\boldsymbol{q}(\boldsymbol{\theta}))\}\right], \qquad \psi(x) := \begin{cases} 2x, & x \leq 1/2, \\ 2x - 1, & x > 1/2. \end{cases} \tag{S1}$$

Equality holds when both buyers' interim utilities at $1/2$ are zero.

*Ironing the revenue bound.* The raw schedule $\psi(x)$ is the critical-type virtual schedule in (12) at $c = 1/2$, with revenue weights $\kappa = 1$ and $w_i = 0$. Its ironed version is

$$\bar{\psi}(x) := \begin{cases} 2x, & 0 \le x < 1/4, \\ 1/2, & 1/4 \le x \le 3/4, \\ 2x - 1, & 3/4 < x \le 1. \end{cases}$$

To verify the direction and equality condition directly, observe that

$$\int_0^x [\psi(s) - \bar{\psi}(s)]\, ds = \begin{cases} 0, & x \in [0, 1/4] \cup [3/4, 1], \\ (x - 1/4)^2, & x \in [1/4, 1/2], \\ (3/4 - x)^2, & x \in [1/2, 3/4]. \end{cases}$$

For every bounded nondecreasing coefficient schedule $m$, Stieltjes integration by parts gives

$$\int_0^1 [\bar{\psi}(x) - \psi(x)] m(x)\, dx = \int_{(0,1)} \left( \int_0^x [\psi(s) - \bar{\psi}(s)]\, ds \right) dm(x) \ge 0.$$

Boundary terms vanish because the inner integral is zero at both endpoints. Equality holds whenever $m$ is constant on $(1/4, 3/4)$. DIC makes each coefficient schedule nondecreasing conditional on the other buyer's report, so applying this inequality to (S1) yields

$$\begin{aligned} \Pi(\boldsymbol{q}, \boldsymbol{t}) &\le \mathbb{E}\left[ \sum_i \{(1 - \bar{\psi}(\theta_i)) q_{i0}(\boldsymbol{\theta}) + \bar{\psi}(\theta_i) q_{i1}(\boldsymbol{\theta})\} \right] \\ &\le \mathbb{E}\left[ \max_{i \in \{1,2\}} \{1 - \bar{\psi}(\theta_i), \bar{\psi}(\theta_i)\} \right]. \end{aligned}$$

The last inequality follows from the single-unit constraint; all these virtual assignment values are nonnegative.

*Constructing an allocation that attains the pointwise bound.* The virtual assignment values are $1 - \bar{\psi}(\theta_i)$ and $\bar{\psi}(\theta_i)$ at locations 0 and 1. If at least one buyer lies outside $[1/4, 3/4]$, the largest value belongs to the buyer farthest from $1/2$, at his preferred location. Assign him the unit, using a fixed priority to break ties between equally distant buyers. If both buyers lie in $[1/4, 3/4]$, all four assignment values equal $1/2$. Set $q^\circ_{i0} = q^\circ_{i1} = 1/4$ for both buyers. This defines a feasible allocation rule $\boldsymbol{q}^\circ$ that exhausts supply and attains the pointwise bound.

*Transfers and incentive compatibility.*

For buyer $i$, write $\boldsymbol{q}^\circ(x, y)$ with $x$ in buyer $i$'s coordinate and $y$ in the other buyer's

coordinate. For fixed $y$, except at the two threshold types, his own-type coefficient is

$$m_i(\boldsymbol{q}^\circ(x,y)) = \begin{cases} -1, & x < 1/2 - \max\{1/4, |y-1/2|\}, \\ 0, & 1/2 - \max\{1/4, |y-1/2|\} < x < 1/2 + \max\{1/4, |y-1/2|\}, \\ 1, & x > 1/2 + \max\{1/4, |y-1/2|\}. \end{cases}$$

The specified tie-breaking gives a coefficient between the adjacent values at each threshold, so the coefficient is nondecreasing at every report. Define truthful utility and transfers by

$$\begin{aligned} U_i^\circ(x,y) &:= \max\{|x-1/2| - \max\{1/4, |y-1/2|\}, 0\}, \\ t_i^\circ(x,y) &:= q_{i0}^\circ(x,y) + x\big[q_{i1}^\circ(x,y) - q_{i0}^\circ(x,y)\big] - U_i^\circ(x,y). \end{aligned}$$

The utility is the integral of the coefficient from $1/2$ to $x$, including at the thresholds by continuity. Monotonicity and the envelope characterization therefore give DIC, and $U_i^\circ \geq 0$ gives EIR. On the central square each buyer pays $1/4$. Outside that square the winner pays $1/2+\max\{1/4, |y-1/2|\}$ and the loser pays zero. Payments are nonnegative at every profile, so EAND holds.

*Attainment and revenue.* The benchmark has $U_i^\circ(1/2, y) = 0$ for every $y$, so the initial utility bound binds. Its coefficient is zero throughout $(1/4, 3/4)$ for every opponent report, so its ironing inequalities also bind. Together with the pointwise maximization established above, these equalities prove revenue optimality over all DIC–IR mechanisms.

For the value of this bound, the random variables $2|\theta_i - 1/2|$, $i = 1, 2$, are independent and uniform on $[0, 1]$, and

$$\max\{1 - \bar{\psi}(\theta_i), \bar{\psi}(\theta_i)\} = \max\{1/2, 2|\theta_i - 1/2|\}.$$

Consequently,

$$\begin{aligned} \Pi(\boldsymbol{q}^\circ, \boldsymbol{t}^\circ) &= \mathbb{E}[\max\{1/2, 2|\theta_1 - 1/2|, 2|\theta_2 - 1/2|\}] \\ &= \frac{1}{2} + \int_{1/2}^{1} (1 - z^2)\, dz = \frac{17}{24}. \end{aligned}$$

*Inefficiency.* On the central square the lottery produces surplus $1/2$, whereas efficient surplus is

$$G(\theta_1, \theta_2) = \frac{1}{2} + \max\{|\theta_1 - 1/2|, |\theta_2 - 1/2|\}.$$

The latter is strictly larger except at the midpoint profile. The central square has probability $1/4$, so the benchmark is not ex post efficient.

*Rent gaps.* The benchmark's profilewise ex post worst-off set for buyer $i$ is $[1/2 -$

$\max\{1/4, |y-1/2|\}, 1/2 + \max\{1/4, |y-1/2|\}]$. Since $\max\{1/4, |y-1/2|\} \geq 1/4$, with equality throughout $y \in [1/4, 3/4]$, its common ex post worst-off set is exactly $[1/4, 3/4]$. Under efficiency, $G(x, y)$ is minimized over $x$ on

$$[1/2 - |y-1/2|,\ 1/2 + |y-1/2|].$$

The efficient accumulated-rent identity (16) therefore makes $1/2$ a common ex post worst-off type under efficiency as well. Both rent gaps are zero by Theorem 1.

*Failure of strict Pareto improvement.* Revenue optimality over DIC–IR rules out any DIC–IR–EAND mechanism that strictly improves all parties' expected payoffs, because the seller cannot strictly gain. Improvements that leave the seller's profit unchanged are not ruled out. □

## S1.2 The Two-Partner Illustration

This online appendix supplies the calculations and implementations used in Section 6. Payoffs and surplus are measured relative to initial ownership throughout. The numerical calculations use independent uniform types unless stated otherwise.

The objectives in Section 6.2 have weights $(w_0, w_1, w_2) = (1, 0, 0), (0, 1/2, 1/2), (0, 1, 0)$, respectively. Their supporting profit weights are $\kappa = 1, 3/4, 1$ under DIC–EIR–EAND and $\kappa = 1, 1/2, 1$ under BIC–IR–EAND.

In the notation of Section 2, $m_i(\boldsymbol{q}) = q_i - r_i$ and $a_i = C = 0$. With $\alpha_i = 1 - w_i/\kappa$, the uniform virtual types are

$$\psi_i^B(x) = (1+\alpha_i)x - \alpha_i, \qquad \psi_i^S(x) = (1+\alpha_i)x.$$

Moving partner $j$'s initial share to partner $i$ yields virtual surplus $r_j[\psi_i^B(\theta_i) - \psi_j^S(\theta_j)]$. At most one trade direction has a positive gain, since $\psi_i^B \leq \theta_i \leq \psi_i^S$. The gain is positive precisely when $\theta_i - \theta_j > 1/2$ for $(w_1, w_2, \kappa) = (0, 0, 1)$, and when $\theta_i - \theta_j > 1/4$ for $(w_1, w_2, \kappa) = (1/2, 1/2, 3/4)$. For $(w_1, w_2, \kappa) = (1, 0, 1)$, the conditions are $\theta_1 > 2\theta_2$ for trade to partner 1 and $2\theta_2 - 1 > \theta_1$ for trade to partner 2. Theorem 2 gives fixed-$\kappa$ optimality of each EIR-normalized mechanism.

To connect this calculation to EAND, let $J = \sum_i w_i \mathbb{E}[U_i] + w_0 \Pi$ and $\kappa = w_0 + \lambda$, with $\lambda \geq 0$. If a fixed-$\kappa$ optimizer $(\boldsymbol{q}^*, \boldsymbol{t}^*)$ satisfies EAND and $\lambda \Pi(\boldsymbol{q}^*, \boldsymbol{t}^*) = 0$, every DIC–EIR–EAND mechanism satisfies

$$J(\boldsymbol{q}, \boldsymbol{t}) \leq W_\kappa(\boldsymbol{q}, \boldsymbol{t}) \leq W_\kappa(\boldsymbol{q}^*, \boldsymbol{t}^*) = J(\boldsymbol{q}^*, \boldsymbol{t}^*).$$

The budget calculations below establish these conditions for the multipliers in Table 1.

| Objective | $(w_0, w_1, w_2)$ | $\kappa$ | No-trade region | $\Pi$ | $(\mathbb{E}[U_1], \mathbb{E}[U_2])$ |
|---|---|---|---|---|---|
| Expected profit | $(1, 0, 0)$ | 1 | $\lvert\theta_1 - \theta_2\rvert \leq 1/2$ | $16/384$ | $(8/384, 8/384)$ |
| Average utility | $(0, 1/2, 1/2)$ | $3/4$ | $\lvert\theta_1 - \theta_2\rvert \leq 1/4$ | 0 | $(27/384, 27/384)$ |
| Partner 1's utility | $(0, 1, 0)$ | 1 | $\theta_1/2 \leq \theta_2 \leq (\theta_1 + 1)/2$ | 0 | $(32/384, 16/384)$ |

Table 1: Optimal DIC–EIR–EAND rules and supporting profit weights, with EIR-normalized expected payoffs. The trading regions and expected payoffs apply at every ownership vector.